DEFORMATIONS OF G-VALUED PSEUDOCHARACTERS

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Abstract. We define a deformation space of V. Lafforgue's G-valued pseudocharacters of a profinite group Γ for a (generalized) reductive group G. We show, that our definition generalizes Chenevier's construction [Che14]. We show that the universal pseudodeformation ring is noetherian whenever Γ is topologically finitely generated. For $G = \mathrm{Sp}_{2n}$ we describe three types of obstructed subloci of the special fiber of the universal pseudodeformation space of an arbitrary residual pseudocharacter and give upper bounds for their dimension.

Contents

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Introduction

Let $F/\mathbb{Q}_p$ be a p-adic local field with absolute Galois group Γ_F. Let L be a p-adic local field with ring of integers $\mathcal{O}_L$ and residue field κ. Let G be a generalized reductive group scheme over $\mathcal{O}_L$ (see Section 1.1), which is essentially a model of a possibly disconnected reductive group over $\mathcal{O}_L$. Given a continuous representation $\overline{\rho} : \Gamma_F \to G(\kappa)$, we define the framed deformation functor on the category $\mathfrak{A}_{\mathcal{O}_L}$ of local artinian $\mathcal{O}_L$-algebras with residue field κ by $\mathrm{Def}^{\square}_{\mathcal{O}_L,\overline{\rho}}(A) := \{\rho : \Gamma_F \to G(A) \mid \rho \text{ continuous lift of } \overline{\rho}\}$. The framed deformation functor is pro-representable by a complete local noetherian $\mathcal{O}_L$-algebra $R^{\square}_{G,\overline{\rho}}$ with residue field κ. Inspired by [BIP21, Theorem 1.1], we would like to prove the following conjecture:

Conjecture 0.1. The ring $R^{\square}_{G,\overline{\rho}}$ is a normal, local complete intersection, flat over $\mathcal{O}_L$ and of relative dimension $\dim G_L \cdot ([F : \mathbb{Q}_p] + 1)$ over $\mathcal{O}_L$.

The proof in [BIP21] relies on estimates of certain subloci in the special fiber of the pseudodeformation ring for GL_n. There pseudocharacters in the sense of Chenevier [Che14] are used.

The first main aim of this paper is to introduce the pseudodeformation ring for generalized reductive group schemes, replacing Chenevier's pseudocharacters by Lafforgue's pseudocharacters as introduced in [Laf18, §11]. We show, that these rings are noetherian for topologically finitely generated profinite groups and in particular for Γ_F.

Theorem A (Theorem 5.9, Theorem 5.12)**.** Let G be a generalized reductive $\mathcal{O}_L$-group scheme, let Γ be a profinite group and let $\overline{\Theta}$ be a continuous G-pseudocharacter of Γ over κ.

(1) If Γ is topologically finitely generated, then the G-pseudodeformation ring $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ of $\overline{\Theta}$ is noetherian.
(2) Assume that $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$, $p > 2$ in the orthogonal cases and let $\iota : G \to \mathrm{GL}_d$ be the standard representation of G. Then the canonical map $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is surjective. If in addition Γ satisfies Mazur's condition Φ_p, then $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is noetherian.

The second main aim is to give estimates for certain obstructed subloci $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$, $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$ and $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ (see Definition 8.3) of the special fiber $\overline{X}_{\overline{\Theta}}$ of $\mathrm{Spec}(R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}})$ analogous to [BIP21, §3.4] and [BJ19], which paves the way for proving Conjecture 0.1 when $G = \mathrm{Sp}_{2n}$.

Theorem B (Proposition 8.11, Theorem 8.12, Corollary 8.13)**.** Let $\overline{\Theta}$ be a continuous Sp_{2n}-pseudocharacter of Γ_F over κ.

(1) $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq n(2n+1)[F : \mathbb{Q}_p] - 4(n-1)[F : \mathbb{Q}_p]$.
(2) $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq n^2[F : \mathbb{Q}_p] + 1$.
(3) $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F : \mathbb{Q}_p] + 1$.
(4) $\dim \overline{X}_{\overline{\Theta}} \leq n(2n+1)[F : \mathbb{Q}_p]$.
If $\overline{\Theta}$ comes from an absolutely irreducible representation, then equality holds and $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \subsetneq \overline{X}_{\overline{\Theta}}$.

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1. Reductive group schemes

1.1. **Generalities.** Working with deformations of representations valued in other algebraic groups G than GL_n, we have to decide which groups we want to allow for G. Our group G shall be naturally defined over the coefficient ring of some deformation problem, for example the ring of integers of a p-adic local field. In [Con14b, Definition 3.1.1] Brian Conrad introduces reductive and semisimple group schemes over arbitrary base schemes.

Definition 1.1. A *reductive (semisimple) group scheme* over a scheme S is a smooth S-affine S-group scheme G, such that the geometric fibers of G are connected reductive (semisimple) groups.

An S-group scheme D is of *multiplicative type*, if it is fppf-locally diagonalizable, i.e. there is an fppf-covering $\{S_i \to S\}$, such that D_{S_i} is isomorphic to the relative spectrum of the quasi-coherent Hopf algebra $\mathcal{O}_{S_i}[M_i]$ for a finitely generated abelian group M_i, where the comultiplication is given by $\Delta(m) = m \otimes m$ and the antipode is given by $s(m) := m^{-1}$ for $m \in M_i$. An *S-torus* is an S-group scheme of multiplicative type with smooth connected fibers.

If G is a reductive S-group scheme, then a *maximal torus* of G is an S-torus $T \subseteq G$, such that for each geometric point $\overline{s}$ of S, $T_{\overline{s}}$ is a maximal torus of $G_{\overline{s}}$. G admits étale-locally a maximal torus [Con14b, Corollary 3.2.7]. For the slightly technical definition of a *split reductive group* over S, we refer to [Con14b, Definition 5.1.1]. If $S = \mathrm{Spec}(\mathbb{Z})$ and G admits a maximal torus, then G is split [Con14b, Example 5.1.4].

Definition 1.2. A *Chevalley group* is a reductive $\mathbb{Z}$-group scheme, which admits a fiberwise maximal $\mathbb{Z}$-torus.

The following three sets are canonically in bijection [Con14a, Theorem 1.4].

(1) Chevalley groups up to $\mathbb{Z}$-isomorphism.
(2) Split connected reductive groups over $\mathbb{Q}$ up to $\mathbb{Q}$-isomorphism.
(3) Root data up to isomorphism.

Every split connected reductive group G over the fraction field K of a domain $\mathcal{O}$ admits a model over $\mathcal{O}$, which is the base change of a Chevalley group over $\mathbb{Z}$ [Con14a, Theorem 1.2]. If $\mathcal{O}$ is a PID, then every $\mathcal{O}$-model of G is the base change of a Chevalley group [Con14a, Proposition 1.3]. We will use these facts to reduce some of our arguments to Chevalley groups. By a *Chevalley group* over another base than $\mathbb{Z}$ we will always mean the base change of a Chevalley group over $\mathbb{Z}$.

Example 1.3. We discuss the main examples we are going to consider.

(1) The symplectic group Sp_{2n} over $\mathbb{Z}$ is the scheme-theoretic automorphism group of the standard symplectic bilinear form on $\mathbb{Z}^{2n}$. Sp_{2n} is a semisimple Chevalley group with almost-simple connected geometric fibers.
(2) The orthogonal group O_n over $\mathbb{Z}[\frac{1}{2}]$ is the automorphism group of the standard symmetric bilinear form on $\mathbb{Z}^n$. O_n is a smooth affine $\mathbb{Z}[\frac{1}{2}]$-group scheme with non-connected almost-simple geometric fibers. SO_n is a reductive $\mathbb{Z}[\frac{1}{2}]$-group scheme.

Suppose G is a connected reductive group over an algebraically closed field k. We recall:

Definition 1.4.

(1) A subgroup H of $G(k)$ is *G-completely reducible* if for every parabolic $P \subseteq G$ with $H \subseteq P(k)$, there exists a Levi subgroup $L \subseteq P$ with $H \subseteq L(k)$.

(2) We say, that a homomorphism $\Gamma \to G(k)$ is *G-completely reducible*, if its image is G-completely reducible.

The definition of G-pseudocharacters Definition 2.1 shall be given in a way that also allows for G to be disconnected. Let G be a smooth affine group scheme over a commutative ring $\mathcal{O}$.

Definition 1.5. We say, that G is a *generalized reductive group scheme*, if G^0 as defined in [Con14b, §3.1] is a reductive group scheme and G/G^0 is finite étale.

The definition of generalized reductive group scheme is given [FM88, Definition 2.1] in terms of a short exact sequence.

1.2. **Topologizing point sets.** Since we work frequently with topologies on point sets of schemes, we want to discuss the general procedure by which all these topologies we are interested in can be obtained. The method is due to Grothendieck and we follow the exposition of [Con12].

Proposition 1.6. Let A be a topological commutative ring. There is a unique way to define a topology on $X(A)$ for all affine A-schemes X of finite type at once, such that the following properties hold.

(1) For every morphism $f : X \to Y$ of affine A-schemes of finite type, the map $X(A) \to Y(A)$ is continuous.
(2) For every cartesian diagram

$$\begin{array}{ccc} X \times_Z Y & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

the diagram of topological spaces

$$\begin{array}{ccc} (X \times_Z Y)(A) & \longrightarrow & X(A) \\ \downarrow & & \downarrow \\ Y(A) & \longrightarrow & Z(A) \end{array}$$

is cartesian.
(3) For every closed immersion $f : X \to Y$ of affine A-schemes of finite type, the map $X(A) \to Y(A)$ is a topological embedding, i.e. $X(A)$ carries the subspace topology of $Y(A)$.
(4) The canonical bijection $A \to \mathbb{A}^1(A)$ is a homeomorphism.

Proof. [Con12, Proposition 2.1]. □

For a finite type affine A-scheme X this topology can be characterized as the coarsest topology on $X(A)$, such that all morphisms of A-schemes $X \to \mathbb{A}^1$ induce a continuous map $X(A) \to A$. It can also be defined by choosing an arbitrary closed immersion $X \to \mathbb{A}^n$ and introducing the subspace topology on $X(A)$ with respect to the injection $X(A) \to A^n$, where A^n carries the product topology.

For every topological commutative A-algebra B, we have $X(B) = X_B(B)$ and we take on $X(B)$ the topology on $X_B(B)$. By choosing an embedding into an affine space, we see that the map $X(B_1) \to X(B_2)$ is continuous for any two topological A-algebras B_1, B_2 and continuous A-homomorphisms $B_1 \to B_2$.

For the proof of Proposition 7.1 we will also need Proposition 1.6 in the following situation: Let κ be a topological field and let A be a finite-dimensional local κ-algebra with residue field κ equipped with the product topology induced by an isomorphism $A \cong \kappa^n$ of κ-vector spaces. If X is an affine A-scheme of finite type, the map $X(A) \to X(\kappa)$ is continuous. If we now take the preimage $Z \subseteq X(A)$ of a Zariski-closed subset $Y(\kappa) \subseteq X(\kappa)$ for some closed A-subscheme $Y \subseteq X$, it is not clear how to identify Z with the A-points of a closed A-subscheme of X, but we still want to describe the topology of Z in a functorial way. This can be done by Weil restriction: The functor $T \mapsto X(A \otimes_\kappa T)$ is representable by an affine κ-scheme $\operatorname{Res}^A_\kappa X$ with $(\operatorname{Res}_{A/\kappa} X)(\kappa) = X(A)$ and the projection $X(A) \to X(\kappa)$ gives rise to a morphism of κ-schemes $\operatorname{Res}_{A/\kappa} X \to X_\kappa$. We now obtain Z as the κ-points of the scheme-theoretic preimage of Y_κ in $\operatorname{Res}_{A/\kappa} X$.

1.3. **Acyclic G-modules and good filtrations.** Let $\mathcal{O}$ be a commutative ring. If V is an $\mathcal{O}$-module with a rational action of an affine $\mathcal{O}$-group scheme G and $\mathcal{O}'$ is an arbitrary commutative $\mathcal{O}$-algebra, then the natural map $V^G \otimes_{\mathcal{O}} \mathcal{O}' \to (V \otimes_{\mathcal{O}} \mathcal{O}')^{G_{\mathcal{O}'}}$ is not always an isomorphism. The entire purpose of this section is to establish conditions under which this map is an isomorphism.

We recall the universal coefficient theorem for rational Ext groups.

Theorem 1.7. Let G be a flat affine group scheme over a Dedekind ring $\mathcal{O}$ and let $\mathcal{O}'$ be a commutative $\mathcal{O}$-algebra. Then for each $\mathcal{O}$-flat G-module N and each finitely generated projective G-module V, we have a short exact sequence

$$0 \to \operatorname{Ext}^n_G(V, N) \otimes_{\mathcal{O}} \mathcal{O}' \to \operatorname{Ext}^n_{G_{\mathcal{O}'}}(V \otimes_{\mathcal{O}} \mathcal{O}', N \otimes_{\mathcal{O}} \mathcal{O}') \to \operatorname{Tor}_1^{\mathcal{O}}(\operatorname{Ext}^{n+1}_G(V, N), \mathcal{O}') \to 0$$

of $\mathcal{O}'$-modules.

Proof. By [Jan03, I.4.4 Lemma] and [Jan03, I.4.2], there is a natural identification $\operatorname{Ext}^n_G(V, N) = \operatorname{Ext}^n_G(\mathcal{O}, V^* \otimes_{\mathcal{O}} N) = H^n(G, V^* \otimes_{\mathcal{O}} N)$ and similarly for the middle term. The claim follows from the universal coefficient theorem [Jan03, I.4.18 Proposition (a)]. □

Corollary 1.8. Let G be a flat affine group scheme over a Dedekind ring $\mathcal{O}$, let V be a G-module and let $\mathcal{O}'$ be a commutative $\mathcal{O}$-algebra. Assume, that one of the following holds:

(1) $\mathcal{O}'$ is $\mathcal{O}$-flat.
(2) $H^1(G, V) = 0$.

Then the natural map $V^G \otimes_{\mathcal{O}} \mathcal{O}' \to (V \otimes_{\mathcal{O}} \mathcal{O}')^{G_{\mathcal{O}'}}$ is an isomorphism.

Proof. By the universal coefficient theorem Theorem 1.7, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}[G^m]^G \otimes_{\mathcal{O}} \mathcal{O}' \longrightarrow \mathcal{O}'[G^m]^G \longrightarrow \operatorname{Tor}_1^{\mathcal{O}}(H^1(G, V), \mathcal{O}') \longrightarrow 0$$

Under both assumptions the claim follows. □

We say, that V is *acyclic*, if the rational cohomology groups $H^i(G, V)$ vanish for all $i > 0$.

If G is a Chevalley group over a principal ideal domain $\mathcal{O}$ with fiberwise maximal $\mathcal{O}$-torus T and Borel subgroup B, we define $H^0(\lambda) := \operatorname{Ind}_B^G \lambda$ and $V(\lambda) := H^0(-w_0\lambda)^*$ for every dominant weight $X(T)_+$ and the longest element w_0 of the Weyl group.

Let V be a G-module. An ascending filtration $V = \bigcup_{i \geq 0} V_i$ of V is *good*, if for all $i \geq 0$, V_{i+1}/V_i is isomorphic to $H^0(\lambda)$ for some $\lambda \in X(T)_+$.

Lemma 1.9. Let G be a Chevalley group over a principal ideal domain $\mathcal{O}$. Let V be a G-module with good filtration. Then V is acyclic.

Proof. If V has finite rank over a PID $\mathcal{O}$, we have $H^i(G,V) = \mathrm{Ext}^i_G(V(0),V) = 0$ for all $i > 0$ by [Jan03, B.9 Lemma (iii)]. If V is not of finite rank, we can choose a good filtration $V = \bigcup_n V_n$ by G-submodules of finite rank and calculate $H^i(G,V) = \varinjlim_n H^i(G,V_n)$ using [Jan03, I.4.17]. □

Mathieu's tensor product theorem states, that the tensor product of two modules with good filtration over a connected reductive group over an algebraically closed field admits a good filtration. An integral version of this theorem also holds and we give a proof here in lack of reference.

Theorem 1.10. Let G be a Chevalley group over a principal ideal domain $\mathcal{O}$. Let M and N be G-modules with good filtration. Then $M \otimes_{\mathcal{O}} N$ is a G-module with good filtration.

Proof. We first assume, that M and N are free of finite rank. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}$ with residue field $\kappa := \mathcal{O}/\mathfrak{m}$. By [Jan03, B.9 Lemma (i) $\Rightarrow$ (iv)], $M_\kappa := M \otimes_{\mathcal{O}} \kappa$ and $N_\kappa := N \otimes_{\mathcal{O}} \kappa$ are G_κ-modules with good filtration. Choose a split and fiberwise maximal $\mathcal{O}$-torus $T \subseteq G$ and a Borel subgroup $B \subseteq G$ containing T. By Theorem 1.7, there is an isomorphism $\mathrm{Ext}^1_{G_\kappa}(V(\lambda), M_\kappa) \otimes_\kappa \overline{\kappa} \cong \mathrm{Ext}^1_{G_{\overline{\kappa}}}(V(\lambda), M_{\overline{\kappa}})$ for all dominant weights $\lambda \in X(T)_+$, so the latter group is 0 by [Jan03, B.9 Lemma (iv) $\Rightarrow$ (iii)] applied to M_κ. So by [Jan03, B.9 Lemma (iii) $\Rightarrow$ (i)] $M_{\overline{\kappa}}$ and $N_{\overline{\kappa}}$ are $G_{\overline{\kappa}}$-modules with good filtration. By Mathieu's tensor product theorem [Mat90], which holds for connected reductive groups over algebraically closed fields, see [Jan03, Proposition II.4.21][vdK93, Theorem 4.4.3] $M_{\overline{\kappa}} \otimes_{\overline{\kappa}} N_{\overline{\kappa}}$ has a good filtration.

We now reverse the argument: By [Jan03, B.9 Lemma (i) $\Rightarrow$ (iii)], we have $\mathrm{Ext}^1_G(V(\lambda), M_{\overline{\kappa}} \otimes_{\overline{\kappa}} N_{\overline{\kappa}})$, hence $\mathrm{Ext}^1_G(V(\lambda), M_\kappa \otimes_\kappa N_\kappa)$ for all dominant weights $\lambda \in X(T)_+$. So by [Jan03, B.9 Lemma (iii) $\Rightarrow$ (i)] $M_\kappa \otimes_\kappa N_\kappa$ has a good filtration. Since $\mathfrak{m}$ is arbitrary, we can apply [Jan03, B.9 Lemma (iv) $\Rightarrow$ (i)] to conclude, that $M \otimes_{\mathcal{O}} N$ is a G-module with good filtration.

Now let M and N be arbitrary with good filtrations $M = \bigcup_{i=1}^\infty M_i$ and $N = \bigcup_{j=1}^\infty N_j$. Then $M \otimes_{\mathcal{O}} N = \bigcup_i \bigcup_j M_i \otimes_{\mathcal{O}} N_i$ by [Sta22, 00DD]. By choosing a diagonal sequence, we can define a filtration of $M \otimes_{\mathcal{O}} N$ by good submodules. □

Proposition 1.11. Let G be a Chevalley group. Then for all $m \geq 1$, $\mathbb{Z}[G^m]$ equipped with the action of G by conjugation has a good filtration. In particular for every commutative ring $\mathcal{O}$ and every $\mathcal{O}$-algebra $\mathcal{O}'$, the canonical map $\mathcal{O}[G^m]^G \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[G^m]^G$ is an isomorphism.

Proof. In [Jan03, B.8] it is shown, that $\mathbb{Z}[G]$ has a good filtration. Here the action of G is defined by $(g \cdot f)(h) := f(g^{-1}hg)$. By Mathieu's tensor product theorem Theorem 1.10, $\mathbb{Z}[G^m] = \mathbb{Z}[G]^{\otimes m}$ has a good filtration. This proves the first assertion. So $H^1(G, \mathbb{Z}[G^m]) = 0$ by Lemma 1.9. We calculate

$$\mathcal{O}[G^m]^G \otimes_{\mathcal{O}} \mathcal{O}' = (\mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathcal{O}) \otimes_{\mathcal{O}} \mathcal{O}' = \mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathcal{O}' = \mathcal{O}'[G^m]^G$$

by applying twice Corollary 1.8. □

Proposition 1.12. For all $m, n \geq 1$, $\mathbb{Z}[M_n^m]$ equipped with the action of $G = \mathrm{GL}_n$ (resp. $G = \mathrm{SL}_n$) by conjugation has a good filtration. In particular for every commutative ring $\mathcal{O}$ and every $\mathcal{O}$-algebra $\mathcal{O}'$, the canonical map $\mathcal{O}[M_n^m]^G \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[M_n^m]^G$ is an isomorphism.

Proof. Let Std be the standard representation of G. Since the $M_n \cong \mathrm{Std} \otimes \mathrm{Std}^*$ and Std is self-dual, we have $M_n^m \cong \mathrm{Std}^{\otimes 2m}$. By Theorem 1.10 and the formula for symmetric powers of direct sums it is enough to show, that $\mathrm{Sym}^d(\mathrm{Std})$ has a good filtration. But $\mathrm{Sym}^d(\mathrm{Std})$ is a highest weight module, so we are done. □

Proposition 1.13. Let $\mathcal{O}$ be a commutative ring with $2 \in \mathcal{O}^\times$ and let $\mathcal{O}'$ be an $\mathcal{O}$-algebra. Then for all $n \geq 0$ the canonical map $\mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{O}_{2n+1}} \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[\mathrm{O}_{2n+1}^m]^{\mathrm{O}_{2n+1}}$ is an isomorphism.

Proof. We have $\mathrm{O}_{2n+1} = \mathrm{SO}_{2n+1} \times \{\pm 1\}$ over $\mathcal{O}$, so we can explicitly compute:

$$\mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{O}_{2n+1}} = \mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{SO}_{2n+1}} = \mathcal{O}[\{\pm 1\}^m] \otimes_{\mathcal{O}} \mathcal{O}[\mathrm{SO}_{2n+1}^m]^{\mathrm{SO}_{2n+1}}$$

We have $\mathcal{O}[\mathrm{SO}_{2n+1}^m]^{\mathrm{SO}_{2n+1}} \otimes_{\mathcal{O}} k = k[\mathrm{SO}_{2n+1}^m]^{\mathrm{SO}_{2n+1}}$ by Proposition 1.11. □

2. G-valued pseudocharacters

Fix a commutative ring $\mathcal{O}$. We will be interested in the case, that $\mathcal{O}$ is the ring of integers of a p-adic field. Let G be a generalized reductive $\mathcal{O}$-group scheme. By the datum of G, the datum of $\mathcal{O}$ is given and we will drop $\mathcal{O}$ from notations. A G-pseudocharacter will be defined depending on both the coefficient ring $\mathcal{O}$ and a commutative $\mathcal{O}$-algebra A, which corresponds to the base ring A in Section 6.1.

2.1. G-pseudocharacters.

The definition of G-pseudocharacter we give is slightly more general than Lafforgue's original definition [Laf18, §11], in that we work over arbitrary base rings $\mathcal{O}$.

We introduce a special notation for substitutions, which will be particularly important in Definition 2.1 and the proofs of Theorem 3.11 and Proposition 3.12.

Let $\mathrm{FG}(m)$ be the free group on m generators $x_1, \dots, x_m$. Let $\alpha : \mathrm{FG}(m) \to \mathrm{FG}(n)$ be a group homomorphism. Let Γ be an arbitrary group. Then there is a unique map $(-)_\alpha : \Gamma^n \to \Gamma^m,\ \gamma \mapsto \gamma_\alpha$, such that the homomorphism $f_\gamma : \mathrm{FG}(n) \to \Gamma,\ x_i \mapsto \gamma_i$ satisfies $f_\gamma(\alpha(x_j)) = (\gamma_\alpha)_j$ for all $j \in \{1, \dots, m\}$. In other words $(-)_\alpha$ is the induced map $\Gamma^n = \mathrm{Hom}(\mathrm{FG}(n), \Gamma) \to \mathrm{Hom}(\mathrm{FG}(m), \Gamma) = \Gamma^m$. More concretely $w_i = \alpha(x_i)$ is a word in x_j and x_j^{-1} for $j = 1, \dots, n$ and α applied to a tuple $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ is the tuple $(\delta_1, \dots, \delta_m) \in \Gamma^m$ with δ_i the word w_i with x_j substituted by γ_j for $j = 1, \dots, n$.

Similarly we obtain an induced map $(-)_\alpha : G^n \to G^m$. G^0 acts on G^m by $g \cdot (g_1, \dots, g_m) = (gg_1g^{-1}, \dots, gg_mg^{-1})$. This induces a rational action of G^0 on the affine coordinate ring $\mathcal{O}[G^m]$ of G^m. The submodule $\mathcal{O}[G^m]^{G^0} \subseteq \mathcal{O}[G^m]$ is defined as the rational invariant module of the G^0-representation $\mathcal{O}[G^m]$. It is an $\mathcal{O}$-subalgebra, since G^0 acts by $\mathcal{O}$-linear automorphisms. The map $(-)_\alpha : G^n \to G^m$ is G^0-equivariant. So there is an induced homomorphism between the algebras of rational invariants $(-)^\alpha : \mathcal{O}[G^m]^{G^0} \to \mathcal{O}[G^n]^{G^0}$. In the special case, that α is induced by a map of sets $\zeta : \{1, \dots, n\} \to \{1, \dots, m\}$, such that $\alpha(x_i) = x_{\zeta(i)}$, we also write $\gamma_\zeta := \gamma_\alpha$ for $\gamma \in \Gamma^n$ and $f^\zeta := f^\alpha$ for $f \in \mathcal{O}[G^m]^{G^0}$.

Definition 2.1 (G-pseudocharacter)**.** Let Γ be a group and let A be a commutative $\mathcal{O}$-algebra. A *G-pseudocharacter* Θ of Γ over A is a sequence of $\mathcal{O}$-algebra maps

$$\Theta_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$$

for each $m \geq 1$, satisfying the following conditions:

(1) For all $n, m \geq 1$, each map $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$, every $f \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma_1, \dots, \gamma_n \in \Gamma$, we have
$$\Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) = \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)})$$
where $f^\zeta(g_1, \dots, g_n) = f(g_{\zeta(1)}, \dots, g_{\zeta(m)})$.

(2) For all $m \geq 1$, for all $\gamma_1, \dots, \gamma_{m+1} \in \Gamma$ and every $f \in \mathcal{O}[G^m]^{G^0}$, we have
$$\Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_{m+1}) = \Theta_m(f)(\gamma_1, \dots, \gamma_m\gamma_{m+1})$$
where $\hat{f}(g_1, \dots, g_{m+1}) = f(g_1, \dots, g_mg_{m+1})$.

We denote the set of G-pseudocharacters of Γ over A by $\mathrm{PC}_G^\Gamma(A)$. If $f : A \to B$ is a homomorphism of $\mathcal{O}$-algebras, then there is an induced map $f_* : \mathrm{PC}_G^\Gamma(A) \to \mathrm{PC}_G^\Gamma(B)$. For $\Theta \in \mathrm{PC}_G^\Gamma(A)$, the image $f_*\Theta$ is called the *scalar extension* of Θ and also denoted with $\Theta \otimes_A B$. This notion of scalar extension shall not be confused with change of the base ring $\mathcal{O}$ of G, which will be discussed in Proposition 3.13 and comes with some subtleties.

If $\iota : G \to H$ is a homomorphism of affine $\mathcal{O}$-group schemes, we define an H-pseudocharacter $\iota(\Theta)$ by letting $\iota(\Theta)_m$ be the composition of Θ_m with the induced map $\mathcal{O}[H^m]^{H^0} \to \mathcal{O}[G^m]^{G^0}$.

In [BHKT16, Def. 4.1] a G-pseudocharacter is defined only for Chevalley groups over $\mathbb{Z}$. Some of our proofs do not need this strong assumption.

Every G-valued representation gives rise to a G-pseudocharacter.

Lemma 2.2. Let Γ be a group, let A be a commutative $\mathcal{O}$-algebra and let $\rho : \Gamma \to G(A)$ be a homomorphism. Then the sequence of maps $\Theta_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$ defined by

$$\Theta_m(f)(\gamma_1, \dots, \gamma_m) := f(\rho(\gamma_1), \dots, \rho(\gamma_m))$$

is a G-pseudocharacter $\Theta = (\Theta_m)_{m \geq 1}$, which depends only on ρ up to $G(A)$-conjugation. We write $\Theta_\rho := \Theta$. In particular the map

$$\begin{aligned} \mathrm{Hom}(\Gamma, G(A))/G^0(A) &\to \mathrm{PC}_G^\Gamma(A) \\ \rho &\mapsto \Theta_\rho \end{aligned}$$

is well-defined, where $G^0(A)$ acts by pointwise conjugation.

Proof. Compare [BHKT16, 4.3]. Let $\gamma_1, \dots, \gamma_n \in \Gamma$ and $\zeta : \{1, \dots, m\} \to \{1, \dots, n\}$ and $f \in \mathcal{O}[G^m]^{G^0}$.

$$\begin{aligned} \Theta_n(f^\zeta)(\gamma_1, \dots, \gamma_n) &= f^\zeta(\rho(\gamma_1), \dots, \rho(\gamma_n)) \\ &= f(\rho(\gamma_{\zeta(1)}), \dots, \rho(\gamma_{\zeta(m)})) \\ &= \Theta_m(f)(\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(m)}) \\ &= \Theta_m(f)^\zeta(\gamma_1, \dots, \gamma_n) \end{aligned}$$

The second property can be checked by a similar calculation. For $g \in G(A)$ and $f \in \mathcal{O}[G^n]^{G^0}$, we have

$$f(g\rho(\gamma_1)g^{-1}, \dots, g\rho(\gamma_n)g^{-1}) = f(\rho(\gamma_1), \dots, \rho(\gamma_n))$$

since f is invariant under conjugation. ☐

Theorem 2.3. Let Γ be a group. Assume that one of the following holds:

(1) G is a Chevalley group over $\mathbb{Z}$ and k is an algebraically closed field.
(2) G is a group scheme over a domain $\mathcal{O}$ of characteristic 0 and k is a field, which contains $\mathcal{O}$, such that G_k is reductive.

Let $\Theta \in \mathrm{PC}_G^\Gamma(k)$. Then there is a finite extension k'/k and a G-completely reducible representation $\rho : \Gamma \to G(k')$ with $\Theta_\rho = \Theta$ and ρ is unique up to $G^0(\overline{k})$-conjugacy.

Proof. The first case is [BHKT16, Theorem 4.5]; we can use Proposition 1.11 to identify the k-points of $G^m /\!\!/ G$ with the k-points of $G_k^m /\!\!/ G_k$. Alternatively we can use [Ses77, Theorem 3]. The second case is [Laf18, Proposition 11.7]. ☐

Remark 2.4. Theorem 2.3 is still true for $G/G^0 \neq 1$ in positive characteristic and can be proved using [Ses77, Theorem 3]. The proof is omitted, as it is not needed for the cases we will consider here.

2.2. **The kernel of a G-pseudocharacter.** It is useful to define the notion of kernel for G-pseudocharacters.

Definition 2.5 (Kernel)**.** Let $\Theta \in \mathrm{PC}_G^\Gamma(A)$ be an arbitrary G-pseudocharacter as in Definition 2.1. We define the *kernel* $\ker(\Theta)$ of Θ as the set of $\delta \in \Gamma$, such that for all $m \geq 1$, all $f \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma_1, \dots, \gamma_m \in \Gamma$, we have

$$\Theta_m(f)(\gamma_1, \dots, \gamma_m \delta) = \Theta_m(f)(\gamma_1, \dots, \gamma_m).$$

Lemma 2.6. $\ker(\Theta)$ in Definition 2.5 is a normal subgroup of Γ.

Proof. It is clear, that $\ker(\Theta)$ is a subgroup of Γ. Let $\delta \in \ker(\Theta)$, $h \in \Gamma$ and $\gamma_1, \dots, \gamma_m \in \Gamma$ for some $m \geq 1$. Then

$$\begin{aligned} \Theta_m(f)(\gamma_1, \dots, \gamma_m h \delta h^{-1}) &= \Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_m h \delta, h^{-1}) \\ &= \Theta_{m+1}(\hat{f})(\gamma_1, \dots, \gamma_m h, h^{-1}) \\ &= \Theta_m(f)(\gamma_1, \dots, \gamma_m) \end{aligned}$$

so $h\delta h^{-1} \in \ker(\Theta)$. ☐

It is easy to check, that if $\delta \in \ker(\Theta)$, then

$$\Theta_m(f)(\gamma_1, \dots, \gamma_{i-1}, \gamma_i \delta, \gamma_{i+1}, \dots, \gamma_m) = \Theta_m(f)(\gamma_1, \dots, \gamma_m)$$

for every $i = 1, \dots, m$.

We will use this to prove the following homomorphisms theorem.

Lemma 2.7. Let $\Theta \in \mathrm{PC}_G^\Gamma(A)$ be an arbitrary G-pseudocharacter as in Definition 2.1, let $\Delta \leq \Gamma$ be a normal subgroup and assume, that $\Delta \subseteq \ker(\Theta)$. Then there is a unique G-pseudocharacter $\Theta' \in \mathrm{PC}_G^{\Gamma/\Delta}(A)$, such that Θ is the restriction of Θ' to Γ.

Proof. Uniqueness is clear, since $\Gamma \to \Gamma/\Delta$ is surjective and hence the maps $\mathrm{Map}((\Gamma/\Delta)^m, A) \to \mathrm{Map}(\Gamma^m, A)$ are injective for all $m \geq 1$. We can define Θ' as

$$\Theta'_m(f)(\gamma_1\Delta, \dots, \gamma_m\Delta) := \Theta_m(f)(\gamma_1, \dots, \gamma_m)$$

for all $m \geq 1$, all $f \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma_1, \dots, \gamma_m \in \Gamma$. This is well-defined, since $\Delta \subseteq \ker(\Theta)$. The axioms of a pseudocharacter are easily verified. ☐

Lemma 2.8. Let $\rho : \Gamma \to G(A)$ be a representation with associated G-pseudocharacter Θ. Then $\ker(\rho) \subseteq \ker(\Theta)$.

Proof. We can define ρ on $\Gamma/\ker(\rho)$. The associated G-pseudocharacter of $\Gamma/\ker(\rho)$ can be inflated to Γ and this turns out to be Θ. ☐

The converse inclusion is false in general! Here is an example.

Example 2.9. Let $\rho : \mathbb{Z} \to \mathrm{GL}_2(\mathbb{C})$ be defined by $\rho(a) := \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$. Then Θ_ρ is the pseudocharacter of the trivial representation. Hence $\ker(\rho) = 1$, but $\ker(\Theta_\rho) = \mathbb{Z}$.

Equality holds, when ρ is G-completely reducible.

Proposition 2.10. Let G be a reductive group over a field k and suppose that one of the assumptions in Theorem 2.3 holds. Let Γ be a group and let $\rho : \Gamma \to G(k)$ be an absolutely G-completely reducible representation with associated G-pseudocharacter Θ. Then $\ker(\rho) = \ker(\Theta)$.

Proof. By Lemma 2.7 Θ factors over a G-pseudocharacter Θ' of $\Gamma/\ker(\Theta)$. By Theorem 2.3 there is a G-completely reducible representation $\rho' : \Gamma/\ker(\Theta) \to G(\overline{k})$ with $\Theta' \otimes_k \overline{k} = \Theta_{\rho'}$. The inflation $\rho'' : \Gamma \to G(\overline{k})$ of ρ' to Γ is still G-completely reducible and conjugate to $\rho \otimes_k \overline{k}$ by an element of $G(\overline{k})$. Hence $\ker(\Theta) \subseteq \ker(\rho'') = \ker(\rho \otimes_k \overline{k}) = \ker(\rho)$. The converse inclusion is Lemma 2.8. □

2.3. **Direct sum, dual and tensor product.** Recall from Section 2.1, that a homomorphism of affine $\mathcal{O}$-group schemes $G \to H$ gives rise to a natural transformation $\mathrm{PC}^{\Gamma}_{G} \to \mathrm{PC}^{\Gamma}_{H}$. This provides us with an astonishingly easy way to define natural operations on pseudocharacters, such as direct sums, duals and tensor products. Defining such operations for determinant laws (see Section 6.1) is more involved; see e.g. [WE13, §1.1.11] for a direct sum operation and [BJ19, §4.5] for twisting with a character. It is clear by construction, that these operations will be compatible with the corresponding operations on representations.

Suppose $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_n}(A)$. Then we can define the *dual* Θ^* by composing with the transpose inverse map $\mathrm{GL}_n \to \mathrm{GL}_n$.

Assume, that $\mathcal{O}$ is a PID. Suppose $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_a}(A)$ and $\Theta' \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_b}(A)$ with $A \in \mathrm{CAlg}_{\mathcal{O}}$ and $a + b = n$. We will define the *direct sum* $\Theta \oplus \Theta' \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_n}(A)$. For $m \geq 1$, we obtain a map $\Theta_m \otimes \Theta'_m : \mathcal{O}[\mathrm{GL}_a^m]^{\mathrm{GL}_a} \otimes_{\mathcal{O}} \mathcal{O}[\mathrm{GL}_b^m]^{\mathrm{GL}_b} \to \mathrm{Map}(\Gamma^m, A)$. It turns out, that since $\mathcal{O}$ is a PID and by Theorem 1.7, we have $\mathcal{O}[\mathrm{GL}_a^m]^{\mathrm{GL}_a} \otimes_{\mathcal{O}} \mathcal{O}[\mathrm{GL}_b^m]^{\mathrm{GL}_b} = \mathcal{O}[(\mathrm{GL}_a \times \mathrm{GL}_b)^m]^{\mathrm{GL}_a \times \mathrm{GL}_b}$. The diagonal embedding $\mathrm{GL}_a \times \mathrm{GL}_b \to \mathrm{GL}_n$ induces a map $\mathcal{O}[\mathrm{GL}_n^m]^{\mathrm{GL}_n} \to \mathcal{O}[(\mathrm{GL}_a \times \mathrm{GL}_b)^m]^{\mathrm{GL}_a \times \mathrm{GL}_b}$ and we define $(\Theta \oplus \Theta')_m$ as the composition of this map with $\Theta_m \otimes \Theta'_m$. The compatibility conditions (1) and (2) in Definition 2.1 can be verified directly, but the alternative description of pseudocharacters Corollary 3.10 in the next section provides us with an easier way to see, that $\Theta \oplus \Theta'$ is indeed a pseudocharacter.

As for the direct sum, the *tensor product* $\Theta \otimes \Theta'$ is induced by the dyadic product map $\mathrm{GL}_a \times \mathrm{GL}_b \to \mathrm{GL}_{ab}$.

We shall also need the notion of direct sum of two symplectic pseudocharacters, induced by the natural map $\mathrm{Sp}_{2a} \times \mathrm{Sp}_{2b} \to \mathrm{Sp}_{2n}$ for $a + b = n$, which corresponds to the orthogonal direct sum of symplectic spaces. The procedure for the construction of this direct sum operation is the same as for the general linear group, explained above.

There is also a natural map $\mathrm{GL}_n \to \mathrm{Sp}_{2n}$ induced by mapping a representation V to $V \oplus V^*$ equipped with the symplectic form, which makes V and V^* totally isotropic subspaces, is the canonical pairing on $V \times V^*$ and the negative of the canonical pairing on $V^* \times V$. Even though the map $\mathrm{GL}_n \to \mathrm{Sp}_{2n}$ is not uniquely determined by this description, it is well-defined on conjugacy classes of representations and well-defined on pseudocharacters.

2.4. **Continuous G-pseudocharacters.** We will also need the notion of a continuous G-pseudocharacter. Assume, that G is an affine group scheme over a commutative ring $\mathcal{O}$.

Definition 2.11 (Continuous G-pseudocharacter)**.** Let Γ be a topological group and let A be a commutative topological $\mathcal{O}$-algebra. A G-pseudocharacter $\Theta \in \mathrm{PC}^{\Gamma}_{\mathcal{O}}(A)$ is *continuous*, if Θ_m takes values in the subset $\mathcal{C}(\Gamma^m, A) \subseteq \mathrm{Map}(\Gamma^m, A)$ of continuous maps for all $m \geq 1$. We write $\mathrm{cPC}^{\Gamma}_{G}(A)$ for the set of continuous G-valued pseudocharacters over A.

It is straightforward to verify, that if G is of finite type over $\mathcal{O}$ and $\rho : \Gamma \to G(A)$ is a continuous homomorphism with $G(A)$ topologized as in Proposition 1.6, then Θ_ρ is a continuous G-pseudocharacter.

3. $\mathcal{C}$-$\mathcal{O}$-algebras

It turns out to be useful to rephrase the definition of G-pseudocharacters in terms of functors on a category $\mathcal{C}$ with values in $\mathcal{O}$-algebras, which we decided to call '$\mathcal{C}$-$\mathcal{O}$-algebras'. Instances of $\mathcal{C}$-$\mathcal{O}$-algebras appear in [Wei20] under the names FI-, FFM- and FFG-algebra. We develop the basic theory of $\mathcal{C}$-$\mathcal{O}$-algebras and use them to prove existence and basic properties of a fine moduli scheme of G-pseudocharacters.

3.1. Generalities.

Definition 3.1 ($\mathcal{C}$-$\mathcal{O}$-algebra)**.** Let $\mathcal{O}$ be a commutative ring and let $\mathcal{C}$ be a small category.

(1) A *$\mathcal{C}$-$\mathcal{O}$-algebra* is a functor
$$\begin{aligned} A^\bullet : \mathcal{C} &\to \mathrm{CAlg}_\mathcal{O} \\ c &\mapsto A^c \end{aligned}$$
into the category of commutative $\mathcal{O}$-algebras $\mathrm{CAlg}_\mathcal{O}$.
(2) A *$\mathcal{C}$-$\mathcal{O}$-homomorphism* between $\mathcal{C}$-$\mathcal{O}$-algebras is a natural transformation $f^\bullet : A^\bullet \to B^\bullet$.
(3) Let $\mathrm{CAlg}_\mathcal{O}^\mathcal{C}$ be the category of $\mathcal{C}$-$\mathcal{O}$-algebras together with $\mathcal{C}$-$\mathcal{O}$-homomorphisms.
(4) A *$\mathcal{C}$-$\mathcal{O}$-subalgebra* of a $\mathcal{C}$-$\mathcal{O}$-algebra $A^\bullet$ is a subfunctor $B^\bullet \subseteq A^\bullet$, such that B^c is an $\mathcal{O}$-subalgebra of A^c for all objects c of $\mathcal{C}$.
(5) A *$\mathcal{C}$-$\mathcal{O}$-ideal* is a subfunctor $I^\bullet \subseteq A^\bullet$, such that I^c is an ideal of A^c for all objects c of $\mathcal{C}$.
(6) A $\mathcal{C}$-$\mathcal{O}$-homomorphism $f^\bullet : A^\bullet \to B^\bullet$ is *injective* (*surjective*, *bijective*) if f^c is injective (surjective, bijective) for all objects c of $\mathcal{C}$.
(7) The *kernel* $\ker(f)^\bullet$ of a $\mathcal{C}$-$\mathcal{O}$-homomorphism $f^\bullet : A^\bullet \to B^\bullet$ is defined by $\ker(f)^c := \ker(f^c)$. It is a $\mathcal{C}$-$\mathcal{O}$-ideal of $A^\bullet$.
(8) The *image* $\mathrm{im}(f)^\bullet$ of a $\mathcal{C}$-$\mathcal{O}$-homomorphism $f^\bullet : A^\bullet \to B^\bullet$ is defined by $\mathrm{im}(f)^c := \mathrm{im}(f^c)$. It is a $\mathcal{C}$-$\mathcal{O}$-subalgebra of $B^\bullet$.

$\mathcal{C}$-$\mathcal{O}$-algebras are just commutative $\mathcal{O}$-algebra objects internal to the topos of $\mathcal{C}$-sets, i.e. functors $\mathcal{C} \to \mathrm{Set}$, and all definitions in Definition 3.1 are valid in this generality.

In universal algebra free algebraic structures can be defined on an arbitrary generating set. Analogously a free $\mathcal{C}$-$\mathcal{O}$-algebra is generated by a $\mathcal{C}$-set. This defines a left adjoint to the forgetful functor $\mathrm{CAlg}_\mathcal{O}^\mathcal{C} \to \mathrm{Set}^\mathcal{C}$. The forgetful functor $\mathrm{Set}^\mathcal{C} \to \mathrm{Set}^{\mathrm{Ob}(\mathcal{C})}$ also admits a left adjoint. Here the set of objects $\mathrm{Ob}(\mathcal{C})$ of $\mathcal{C}$ is regarded as a discrete category and an $\mathrm{Set}^{\mathrm{Ob}(\mathcal{C})}$ is the same as a family of sets indexed by $\mathrm{Ob}(\mathcal{C})$. Since we are only interested in free $\mathcal{C}$-$\mathcal{O}$-algebras on an $\mathrm{Set}^{\mathrm{Ob}(\mathcal{C})}$, we define the composition of these two left adjoints directly.

Lemma 3.2 (Free commutative $\mathcal{C}$-$\mathcal{O}$-algebra)**.** Let $\mathcal{O}$ be a commutative ring, $\mathcal{C}$ a small category and $T^\bullet$ an $\mathrm{Ob}(\mathcal{C})$-set. Then there is a $\mathcal{C}$-$\mathcal{O}$-algebra $F^\bullet$ together with a map of $\mathrm{Ob}(\mathcal{C})$-sets $\iota : T^\bullet \to F^\bullet$, that satisfies the following universal property:

For every map of $\mathrm{Ob}(\mathcal{C})$-sets $f : T^\bullet \to R^\bullet$ to a $\mathcal{C}$-$\mathcal{O}$-algebra $R^\bullet$, there is a unique homomorphism of $\mathcal{C}$-$\mathcal{O}$-algebras $\bar{f} : F^\bullet \to R^\bullet$, such that $\bar{f} \circ \iota = f$. We call the pair $(F^\bullet, \iota)$ the *free $\mathcal{C}$-$\mathcal{O}$-algebra* on $T^\bullet$. It is unique up to unique isomorphism.

Proof. Let $x \in \mathrm{Ob}(\mathcal{C})$. We define F^x to be the free commutative $\mathcal{O}$-algebra generated by the set

$$\coprod_{y \in \mathcal{C}} \coprod_{\alpha \in \mathrm{Hom}_{\mathcal{C}}(y,x)} T^y$$

For $\alpha : y \to x$ we denote the generator of F^x associated to $t \in T^y$ by ${}^{\alpha}t$. Define $\iota^x : T^x \to F^x, t \mapsto {}^{\mathrm{id}_x}t$ to be the inclusion of T^x into the summand associated to id_x. Define for every morphism $\alpha : x \to y$ of $\mathcal{C}$ an $\mathcal{O}$-homomorphism $\alpha_* : T^x \to T^y,\ {}^{\beta}t \mapsto {}^{\alpha\beta}t$. Now let $f : T^\bullet \to R^\bullet$ be a map of $\mathrm{Ob}(\mathcal{C})$-sets. We define for all $x \in \mathcal{C}$ an $\mathcal{O}$-algebra homomorphism $\bar{f}^x : F^x \to R^x,\ {}^{\beta}t \mapsto \beta_*(f^y(t))$, where $\beta : y \to x$ is a morphism of $\mathcal{C}$ and $t \in T^y \subset F^y$. One easily checks $\bar{f} \circ \iota = f$ and this equation forces uniqueness of $\bar{f}$. By the standard argument $(F^\bullet, \iota)$ is unique up to unique isomorphism. □

3.2. **G-pseudocharacters as $\mathcal{F}$-$\mathcal{O}$-algebra homomorphisms.** From now on, we will consider two different small categories for $\mathcal{C}$.

(1) Let $\mathcal{M}$ be the category of free monoids $\mathrm{FM}(m)$ on m generators for all $m \geq 1$.
(2) Let $\mathcal{F}$ be the category of free groups $\mathrm{FG}(m)$ on m generators for all $m \geq 1$.

A monoid homomorphism between finitely generated free monoids can be understood as a finite sequence of words. Such a sequence also defines a homomorphism between free groups and so we get a canonical functor

$$\mathcal{M} \to \mathcal{F}$$

In particular every $\mathcal{F}$-$\mathcal{O}$-algebra can be restricted to an $\mathcal{M}$-$\mathcal{O}$-algebra.

Example 3.3. Here are the two examples of $\mathcal{F}$-$\mathcal{O}$-algebras we are interested in.

(1) If A is an $\mathcal{O}$-algebra, then the functor

$$\mathcal{F} \to \mathrm{CAlg}_{\mathcal{O}},$$
$$\mathrm{FG}(m) \mapsto \mathrm{Map}(\Gamma^m, A)$$

where $\alpha : \mathrm{FG}(n) \to \mathrm{FG}(m)$ is mapped to

$$\alpha_* : \mathrm{Map}(\Gamma^n, A) \to \mathrm{Map}(\Gamma^m, A)$$

where $\alpha_*(f)(\gamma_1, \dots, \gamma_m) := f(\phi(\alpha(x_1)), \dots, \phi(\alpha(x_n)))$, where $\phi : \mathrm{FG}(m) \to \Gamma,\ x_i \mapsto \gamma_i$, defines an $\mathcal{F}$-$\mathcal{O}$-algebra $\mathrm{Map}(\Gamma^\bullet, A)$.
(2) Similarly

$$\mathcal{F} \to \mathrm{CAlg}_{\mathcal{O}}$$
$$\mathrm{FG}(m) \mapsto \mathcal{O}[G^m]^{G^0}$$

defines an $\mathcal{F}$-$\mathcal{O}$-algebra: Every homomorphism $\alpha : \mathrm{FG}(n) \to \mathrm{FG}(m)$ induces a morphism of $\mathcal{O}$-schemes $G^m \to G^n$, which in turn induces the desired map $\alpha_* : \mathcal{O}[G^n]^{G^0} \to \mathcal{O}[G^m]^{G^0}$. Note, that since $G^m \to G^n$ is induced by a homomorphism of free groups it is equivariant with respect to diagonal conjugation and hence α_* is well-defined. We will denote this $\mathcal{F}$-$\mathcal{O}$-algebra by $\mathcal{O}[G^\bullet]^{G^0}$.

By definition a G-pseudocharacter Θ is a sequence of maps $\Theta_m : \mathcal{O}[G^m]^{G^0} \to \mathrm{Map}(\Gamma^m, A)$, that behaves natural with respect to two specified types of monoid homomorphisms. Our next goal is to understand, that these types of monoid homomorphisms do already generate all morphisms in $\mathcal{M}$ and make $\Theta_\bullet = (\Theta_m)_{m \geq 0}$ an $\mathcal{M}$-$\mathcal{O}$-homomorphism. We start with generalities on generating sets of morphisms in categories.

Definition 3.4. Let $\mathcal{C}$ be a category and S a system of morphisms $S_{A,B} \subseteq \mathrm{Hom}_{\mathcal{C}}(A,B)$ for all pairs of objects A, B. Let $\tilde{S}$ be another such system of morphisms.

(1) S *generates* $\tilde{S}$, if $\tilde{S}$ is the smallest system of morphisms, that contains S, all identities and for any two composable morphisms $\alpha_1, \alpha_2 \in \tilde{S}$ their composition $\alpha_2 \circ \alpha_1$.
(2) S *inv-generates* $\tilde{S}$, if $\tilde{S}$ is the smallest system of morphisms, that contains S, all identities, for any two composable morphisms $\alpha_1, \alpha_2 \in \tilde{S}$ their composition $\alpha_2 \circ \alpha_1$ and for each invertible morphism $\alpha \in \tilde{S}$ its inverse α^{-1}.

Remark 3.5. A system of morphisms S always (inv-)generates a unique system of morphisms, since the conditions in Definition 3.4 are closed under arbitrary intersections. If S generates $\tilde{S}$, then $\tilde{S}$ consists of compositions of morphisms of S and identities. If S inv-generates $\tilde{S}$, then $\tilde{S}$ consists of iterated compositions and inversions of morphisms of S that are invertible in $\mathcal{C}$ and identities.

It is enough to check naturality on (inv-)generating systems of morphisms.

Lemma 3.6. Let $\mathcal{C}$ be a category and S a generating system of the morphisms of $\mathcal{C}$. Let $\mathcal{D}$ be another category, $F, G : \mathcal{C} \to \mathcal{D}$ functors and $\Theta : F \to G$ an *infranatural transformation*, i.e. a collection of morphisms $\Theta_X : FX \to GX$ for each object X of $\mathcal{C}$. Further assume, that Θ is natural for morphisms of $\alpha \in S$, i.e. for all $\alpha \in S$ the diagram

$$\begin{array}{ccc} FX & \xrightarrow{\Theta_X} & GX \\ \downarrow{\scriptstyle F\alpha} & & \downarrow{\scriptstyle G\alpha} \\ FY & \xrightarrow{\Theta_Y} & GY \end{array}$$

commutes. Then Θ is a natural transformation. The same is true if S is an inv-generating system.

Proof. Follows easily from Remark 3.5 and structural induction. □

We now determine generating sets of morphisms for $\mathcal{M}$ and $\mathcal{F}$.

Lemma 3.7. The morphisms of $\mathcal{M}$ are generated by the following two types of homomorphisms:

(1) $\phi : \mathrm{FM}(n) \to \mathrm{FM}(m)$, where $\phi(x_i) := x_{\zeta(i)}$ for each map $\zeta : \{1, \dots, n\} \to \{1, \dots, m\}$.
(2) $\phi : \mathrm{FM}(n) \to \mathrm{FM}(n+1)$ where $\phi(x_i) := x_i$ for all $i < n$ and $\phi(x_n) := x_n x_{n+1}$.

The morphisms of $\mathcal{F}$ are generated by homomorphisms of types (1) and (2) with FM replaced by FG and a third type of homomorphism:

(3) $\phi : \mathrm{FG}(n) \to \mathrm{FG}(n)$ where $\phi(x_i) := x_i$ for all $i < n$ and $\phi(x_n) := x_n^{-1}$.

Proof. This is [Wei20, Lem. 2.1]. Let $\phi : \mathrm{FM}(n) \to \mathrm{FM}(m)$ be a homomorphism with $\phi(x_i) = x_{k_{i1}} \dots x_{k_{il_i}}$. We may write it as a tuple

$$(x_{k_{11}} \dots x_{k_{1l_1}}, \dots, x_{k_{n1}} \dots x_{k_{nl_n}})$$

It is a composition of homomorphisms of type (2) and the homomorphism given by

$$(x_{k_{11}} \dots x_{k_{1l_1}}, \dots, x_{k_{(n-1)1}} \dots x_{k_{(n-1)l_{n-1}}}, x_{k_{n1}}, \dots, x_{k_{nl_n}})$$

Iterated application of permutations of type (1) and homomorphisms of type (2) reduces us to

$$(x_{e_1}, \dots, x_{e_t})$$

where $x_{e_1}, \dots, x_{e_t}$ is the condensed sequence of letters $x_{k_{i1}}, \dots, x_{k_{il_i}}$ for all i. This is already a homomorphism of type (1).

For a homomorphism $\Phi : \mathrm{FG}(n) \to \mathrm{FG}(m)$ one analogously reduces using types (1) and (2) to a sequence

$$(x_{e_1}^{\pm 1}, \dots, x_{e_t}^{\pm 1})$$

Application of permutations and homomorphisms of type (3) reduces us to $(x_{e_1}, \dots, x_{e_t})$. □

Proposition 3.8. We have a canonical bijection between the set of $\mathcal{M}$-$\mathcal{O}$-algebra homomorphisms $\mathcal{O}[G^\bullet]^{G^0} \to \mathrm{Map}(\Gamma^\bullet, A)$ and the set $\mathrm{PC}_G^\Gamma(A)$ of G-pseudocharacters of Γ with values in A.

Proof. We start with a G-pseudocharacter $(\Theta_m)_{m\geq 1}$ and define an association $\tilde{\Theta} : \mathcal{O}[G^\bullet]^{G^0} \to \mathrm{Map}(\Gamma^\bullet, A)$ by setting $\tilde{\Theta}_{\mathrm{FM}(m)} := \Theta_m$. By definition of Θ we know, that $\tilde{\Theta}$ is natural with respect to morphisms $\mathrm{FM}(n) \to \mathrm{FM}(m)$ of type (1) and morphisms $\mathrm{FM}(n) \to \mathrm{FM}(n+1)$ of type (2). By Lemma 3.6 and Lemma 3.7 this implies naturality. Conversely, given a morphism $\tilde{\Theta}$ of $\mathcal{M}$-$\mathcal{O}$-algebras, the associated sequence of algebra maps $\Theta_n := \Theta_{\mathrm{FM}(n)}$ satisfies the required properties by naturality. □

Clearly the morphisms of $\mathcal{F}$ are not generated by homomorphisms of type (1) and (2) with FM replaced by FG: Homomorphisms of type (1) and (2) have the property, that the image of the generators x_i lies in the submonoid spanned by generators. This property is stable under compositions and hence the homomorphism $\mathrm{FG}(1) \to \mathrm{FG}(1),\ x_1 \mapsto x_1^{-1}$ is not a composition of type (1) or (2) homomorphisms. Fortunately by Lemma 3.6 we only need, that the morphisms of $\mathcal{F}$ are inv-generated by homomorphisms of type (1) and (2) to prove, that any pseudocharacter gives rise to an $\mathcal{F}$-$\mathcal{O}$-algebra homomorphism.

Lemma 3.9. The morphisms of $\mathcal{F}$ are inv-generated by homomorphisms of type (1) and (2) in Lemma 3.7 with FM replaced by FG.

Proof. By Remark 3.5 and Lemma 3.7 it suffices to show, that homomorphisms of type (3) can be written as iterated compositions and inversions of homomorphisms of type (1) and (2). Since by Lemma 3.7 $\mathcal{M}$ is generated by monoid homomorphisms of types (1) and (2), we already know, that all group homomorphisms $\mathrm{FG}(n) \to \mathrm{FG}(m)$, that are induced by monoid homomorphisms $\mathrm{FM}(n) \to \mathrm{FM}(m)$ are generated by group homomorphisms of type (1) and (2).

Let $\phi : \mathrm{FG}(n) \to \mathrm{FG}(n)$ be of type (3). We will use tuple notation for homomorphisms, so $\phi = (\phi(x_1), \dots, \phi(x_n)) = (x_1, \dots, x_{n-1}, x_n^{-1})$. We first give the proof for $n = 2$ and $x_1 = x, x_2 = y$:

$$(x, y^{-1}) = (xy^{-1}, y) \circ (xy, x) \circ (x, x^{-1}y)$$

Since $(xy^{-1}, y) = (xy, y)^{-1}$ and $(x, x^{-1}y) = (x, xy)^{-1}$ we have shown, that ϕ is inv-generated by homomorphisms of type (1) and (2). This argument works analogously for $n \geq 2$. For $n = 1$ we consider the homomorphisms $(y) : \mathrm{FG}(1) \to \mathrm{FG}(2),\ x \mapsto y$ and $(1, x) : \mathrm{FG}(2) \to \mathrm{FG}(1),\ x \mapsto 1,\ y \mapsto x$ and write $(x^{-1}) = (1, x) \circ (x, y^{-1}) \circ (y)$. □

Corollary 3.10. The maps

$$\begin{aligned}\mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}}}(\mathcal{O}[G^\bullet]^{G^0},\ \mathrm{Map}(\Gamma^\bullet, A)) &\to \mathrm{PC}_G^\Gamma(A) \\ \Theta &\mapsto (\Theta_m)_{m\geq 1}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}}}(\mathcal{O}[G^\bullet]^{G^0},\ \mathcal{C}(\Gamma^\bullet, A)) &\to \mathrm{cPC}_G^\Gamma(A) \\ \Theta &\mapsto (\Theta_m)_{m\geq 1}\end{aligned}$$

are bijections.

Proof. The proof of Proposition 3.8 carries over. □

3.3. Representability of PC_G^Γ.

Theorem 3.11 (Representability of PC_G^Γ)**.** Let Γ be a group and let G be a generalized reductive $\mathcal{O}$-group scheme. The functor $\mathrm{PC}_G^\Gamma : \mathrm{CAlg}_{\mathcal{O}} \to \mathrm{Set}$ is representable by a commutative $\mathcal{O}$-algebra B_G^Γ. There is a universal G-pseudocharacter $\Theta^u \in \mathrm{PC}_G^\Gamma(B_G^\Gamma)$, i.e. for all $m \in \mathbb{N}$, $\mu \in \mathcal{O}[G^m]^G$, $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma^m$, for every $A \in \mathrm{CAlg}_{\mathcal{O}}$ and every $\Theta \in \mathrm{PC}_G^\Gamma(A)$, the associated homomorphism $f_\Theta : B_G^\Gamma \to A$ satisfies $f_\Theta(\Theta_m^u(\mu)(\gamma)) = \Theta_m(\mu)(\gamma)$. As an $\mathcal{O}$-algebra B_G^Γ is generated by $\{\Theta_m^u(\mu)(\gamma) \mid \mu \in \mathcal{O}[G^m]^G,\ \gamma \in \Gamma^m\}$.

In the proof, we only need that G is affine.

Proof. Let $F := \mathcal{O}[\tilde{t}_{\mu,\gamma} \mid m \in \mathbb{N},\ \mu \in \mathcal{O}[G^m]^G,\ \gamma \in \Gamma^m]$ be the free commutative $\mathcal{O}$-algebra generated by the letters $\tilde{t}_{\mu,\gamma}$ for all $m \in \mathbb{N}$, $\mu \in \mathcal{O}[G^m]^G$ and Γ^m. For all $A \in \mathrm{CAlg}_{\mathcal{O}}$ and all $\Theta \in \mathrm{PC}_G^\Gamma(A)$, there is an $\mathcal{O}$-linear map $\tilde{\eta}_\Theta : F \to A,\ \tilde{t}_{\mu,\gamma} \mapsto \Theta_m(\mu)(\gamma)$. Let $\mathfrak{a} \subseteq F$ be the intersection of $\ker(\tilde{\eta}_\Theta)$ for all $A \in \mathrm{CAlg}_{\mathcal{O}}$ and all $\Theta \in \mathrm{PC}_G^\Gamma(A)$. Define $B_G^\Gamma := F/\mathfrak{a}$.

From now on, we let $\eta_\Theta : B_G^\Gamma \to A$ be $\eta_\Theta(x + \mathfrak{a}) := \tilde{\eta}_\Theta(x)$ and $t_{\mu,\gamma} := \tilde{t}_{\mu,\gamma} + \mathfrak{a} \in B_G^\Gamma$. In particular $\eta_\Theta(t_{\mu,\gamma}) = \Theta_m(\mu)(\gamma)$.

For every $A \in \mathrm{CAlg}_{\mathcal{O}}$, we have a map $H_A : \mathrm{PC}_G^\Gamma(A) \to \mathrm{Hom}_{\mathcal{O}}(B_G^\Gamma, A),\ \Theta \mapsto \eta_\Theta$ and these are natural in A. We define the universal pseudocharacter $\Theta^u : \mathcal{O}[G^\bullet]^{G^0} \to \mathcal{C}(\Gamma^\bullet, B_G^\Gamma)$ by $\Theta_m^u(\mu) : \Gamma^m \to B_G^\Gamma,\ \gamma \mapsto t_{\mu,\gamma}$.

We check property (1) in the definition of pseudocharacter for Θ^u, property (2) is similar. Let $\mu \in \mathcal{O}[G^m]^G$ and let $\zeta : \{1, \dots, n\} \to \{1, \dots, m\}$ be some map. Then $\Theta_n^u(\mu^\zeta)(\gamma) = t_{\mu^\zeta,\gamma}$ and $\Theta_m^u(\mu)(\gamma_\zeta) = t_{\mu,\gamma_\zeta}$. Here we write $\gamma_\zeta = (\gamma_{\zeta(1)}, \dots, \gamma_{\zeta(n)})$. We claim, that $t_{\mu^\zeta,\gamma} = t_{\mu,\gamma_\zeta}$. Indeed for every pseudocharacter Θ, we have $\tilde{\eta}_\Theta(\tilde{t}_{\mu^\zeta,\gamma}) = \Theta_n(\mu^\zeta)(\gamma) = \Theta_m(\mu)(\gamma_\zeta) = \tilde{\eta}_\Theta(\tilde{t}_{\mu,\gamma_\zeta})$. So $\tilde{t}_{\mu^\zeta,\gamma} - \tilde{t}_{\mu,\gamma_\zeta} \in \ker(\tilde{\eta}_\Theta)$ and the claim follows by definition of $\mathfrak{a}$.

We see, that for any pseudocharacter Θ, we have $\Theta = \eta_{\Theta *}\Theta^u$ and for every $h \in \mathrm{Hom}_{\mathcal{O}}(B_G^\Gamma, A)$, we have $\eta_{h_*\Theta^u} = h$, so universality of Θ^u and bijectivity of the transformation H follows. □

At this point, we would like to give also a purely categorical proof of Theorem 3.11, which is already inherent in [Zhu20, Remark 2.2.5]. To us the derived perspective is not relevant.

Categorical proof. We use the description of pseudocharacters as $\mathcal{F}$-$\mathcal{O}$-algebra homomorphisms according to Corollary 3.10. We denote by $\mathcal{F}/\Gamma$ the slice category of objects of $\mathcal{F}$ with a fixed homomorphism to Γ. Let

$$B_G^\Gamma := \operatorname*{colim}_{\mathrm{FG}(m) \in \mathcal{F}/\Gamma} \mathcal{O}[G^m]^{G^0}$$

be the colimit in the category of commutative $\mathcal{O}$-algebras indexed over the small category $\mathcal{F}/\Gamma$. Then

$$\begin{aligned}\mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}}(B_G^\Gamma, A) &= \lim_{\mathrm{FG}(m) \in \mathcal{F}/\Gamma} \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}}(\mathcal{O}[G^m]^{G^0}, A)\\ &= \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}/\Gamma}}(\mathcal{O}[G^\bullet]^{G^0}, A)\\ &= \mathrm{Hom}_{\mathrm{CAlg}_{\mathcal{O}}^{\mathcal{F}}}(\mathcal{O}[G^\bullet]^{G^0}, \mathrm{Map}(\Gamma^\bullet, A))\end{aligned}$$

for every $A \in \mathrm{CAlg}_{\mathcal{O}}$, where in the second line A is understood as the constant functor on $\mathcal{F}/\Gamma$. In the last line, we compute the right Kan extension of $A : \mathcal{F}/\Gamma \to \mathrm{CAlg}_{\mathcal{O}}$ along the canonical restriction $p : \mathcal{F}/\Gamma \to \mathcal{F}$ as

$$(\mathrm{Ran}_p\, A)(\mathrm{FG}(m)) = \lim_{\substack{(\mathrm{FG}(n),f)\in\mathcal{F}/\Gamma \\ \varphi\in\mathrm{Hom}(\mathrm{FG}(m),\mathrm{FG}(n))}} A(\mathrm{FG}(n), f) = \lim_{(\mathrm{FG}(n),f)\in\mathcal{F}/\Gamma} \mathrm{Map}(\mathrm{FG}(n)^m, A) = \mathrm{Map}(\Gamma^m, A)$$

using the description of Kan extensions as weighted limits. □

From the categorical proof of Theorem 3.11 it is also clear, that B_G^Γ is finitely generated if Γ and all $\mathcal{O}[G^m]^{G^0}$ are finitely generated. We again give an elementary and a categorical proof.

Proposition 3.12. Let Γ be a finitely generated group, let $\mathcal{O}$ be a commutative ring, which is of finite type over a universally Japanese ring and let G be a generalized reductive $\mathcal{O}$-group scheme. Then B_G^Γ is a finitely generated $\mathcal{O}$-algebra.

Proof. Let r be a number of generators of Γ. We know from [Ses77, Theorem 2 (i)], that $\mathcal{O}[G^r]^{G^0}$ is a finitely generated $\mathcal{O}$-algebra. Let k be a number of $\mathcal{O}$-algebra generators of $\mathcal{O}[G^r]^{G^0}$. Let $s = (s_1, \dots, s_r) \in \Gamma^r$ be generators of Γ and let $f_1, \dots, f_k$ be generators of $\mathcal{O}[G^r]^{G^0}$. With notation as in Theorem 3.11 we claim, that $\{t_{f_i,s} \mid i \in \{1, \dots, k\}\}$ is a system of generators of B_G^Γ. By Theorem 3.11 B_G^Γ is generated by the elements $t_{\mu,\gamma}$ for all $m \geq 1$, all $\mu \in \mathcal{O}[G^m]^{G^0}$ and all $\gamma \in \Gamma^m$. These elements satisfy functoriality properties similar to that of pseudocharacters with respect to the category $\mathcal{F}$, as explained in the proof of Theorem 3.11. Let us fix such an element $t_{\mu,\gamma}$. Every element $\gamma_1, \dots, \gamma_m$ can be written as a product of elements $s_1, \dots, s_r$ and such a presentation determines a homomorphism of free groups $\alpha : \mathrm{FG}(m) \to \mathrm{FG}(r)$, such that the composition with the projection $\mathrm{FG}(r) \twoheadrightarrow \Gamma,\ x_i \mapsto s_i$ maps x_i to γ_i. We have $\gamma = s_\alpha$, so $t_{\mu,\gamma} = t_{\mu,s_\alpha} = t_{\mu^\alpha,s}$. By uniqueness and the defining property, we see, that $t_{\cdot,s} : \mathcal{O}[G^r]^{G^0} \to B_G^\Gamma$ is a homomorphism and it follows, that $t_{\mu^\alpha,s}$ is a product of elements $t_{f_i,s}$. □

Categorical proof. We use the description of B_G^Γ as a colimit as in the categorical proof of Theorem 3.11. If Γ is finitely generated, then $\mathcal{F}/\Gamma$ contains a surjection $\pi : \mathrm{FG}(m) \twoheadrightarrow \Gamma$. For every $\mathrm{FG}(n) \in \mathcal{F}$ every homomorphism $f : \mathrm{FG}(n) \to \Gamma$ factors over π, so the associated map $f_* : \mathcal{O}[G^n]^{G^0} \to B_G^\Gamma$ factors over the map $\pi_* : \mathcal{O}[G^m]^{G^0} \to B_G^\Gamma$ associated to π, which implies, that π_* is surjective. So it suffices, that $\mathcal{O}[G^m]^{G^0}$ is finitely generated. It follows from [Ses77, Theorem 2 (i)], that $\mathcal{O}[G^m]^{G^0}$ is a finitely generated $\mathcal{O}$-algebra. □

Proposition 3.13. Let $\mathcal{O} \to \mathcal{O}'$ be a ring homomorphism, let Γ be a group, let G be a generalized reductive $\mathcal{O}$-group scheme and assume that one of the following holds.

(1) $\mathcal{O}'$ is $\mathcal{O}$-flat.
(2) G is a Chevalley group.
(3) $G = \mathrm{O}_{2n+1}$ and $2 \in \mathcal{O}^\times$.

Then for any $\mathcal{O}'$-algebra A, there is a canonical bijection

$$\mathrm{PC}^\Gamma_{G_{\mathcal{O}'}}(A) \cong \mathrm{PC}^\Gamma_G(A) \tag{3.1}$$

induced by a canonical isomorphism $\mathcal{O}[G^\bullet]^{G^0} \otimes_{\mathcal{O}} \mathcal{O}' \to \mathcal{O}'[G^\bullet]^{G^0}$ of $\mathcal{F}$-$\mathcal{O}'$-algebras. Moreover, there is a canonical isomorphism $B_G^\Gamma \otimes_{\mathcal{O}} \mathcal{O}' \cong B^\Gamma_{G_{\mathcal{O}'}}$ of $\mathcal{O}'$-algebras.

Proof. By Corollary 3.10 it is enough to show, that $\mathcal{O}[G^m]^{G^0} \otimes_{\mathcal{O}} \mathcal{O}' = \mathcal{O}'[G^m]^{G^0}$ for all $m \geq 1$. In the three cases this follows from Corollary 1.8, Proposition 1.11 and Proposition 1.13 respectively.

We now prove that $B_G^\Gamma \otimes_{\mathcal{O}} \mathcal{O}' \cong B_{G_{\mathcal{O}'}}^\Gamma$. We apply Theorem 3.11 twice and the first assertion once: Let A be an $\mathcal{O}'$-algebra.

$$\mathrm{Hom}_{\mathcal{O}'}(B_{G_{\mathcal{O}'}}^\Gamma, A) \overset{3.11}{=} \mathrm{PC}_{G_{\mathcal{O}'}}^\Gamma(A) \overset{(3.1)}{\cong} \mathrm{PC}_G^\Gamma(A) \overset{3.11}{=} \mathrm{Hom}_{\mathcal{O}}(B_G^\Gamma, A) = \mathrm{Hom}_{\mathcal{O}'}(B_G^\Gamma \otimes_{\mathcal{O}} \mathcal{O}', A)$$

The claim follows by Yoneda. □

4. Invariant theory

The goal of this section is to prove, that the $\mathcal{F}$-$\mathbb{Z}/p^r$-algebras $\mathbb{Z}/p^r[G^\bullet]^{G^0}$ are finitely generated for $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_{2n+1}\}$ and to determine an explicit set of generators. We use a theorem of Zubkov [Zub99] on generators of certain invariant rings over an algebraically closed field and generalize it to $\mathbb{Z}/p^r$-algebras.

Let $\mathbb{X} \in M_d(\mathbb{Z}[x_{ij} \mid i,j \in \{1,\dots,d\}])$ be a generic $d \times d$ matrix, i.e. $\mathbb{X}_{ij} = x_{ij}$ for $1 \le i,j \le d$.

Let $s_i \in \mathbb{Z}[x_{ij}]$ be up to a sign the i-th coefficient of the characteristic polynomial of $\mathbb{X}$:

$$\det(t \cdot I_d - \mathbb{X}) = \sum_{i=0}^{d} (-1)^i s_i(\mathbb{X}) t^{d-i} \quad \in \mathbb{Z}[x_{ij} \mid i,j \in \{1,\dots,d\}][t]$$

If we evaluate $\mathbb{X}$ at a triangular matrix, then s_i is given by the i-th elementary symmetric polynomial in the diagonal entries.

Theorem 4.1 (Zubkov, 1999)**.** Let K be an algebraically closed field. Let G be either the symplectic group $\mathrm{Sp}_{d,K}$ for even $d \ge 2$ or the orthogonal group $\mathrm{O}_{d,K}$ for $d \ge 1$ and assume, that $\mathrm{char}(K) \ne 2$ in the orthogonal case. Let $m \ge 1$. The algebraic group G acts by conjugation on the affine K-scheme $M_d^m \cong \mathbb{A}^{md^2}$. Denote by $K[M_d^m]^G$ the algebra of rational invariants of the coordinate ring $K[M_d^m]$ of M_d^m. Denote by $X_k \in M_d(\mathbb{Z}[x_{ij}^{(k)} \mid i,j \in \{1,\dots,d\}])$ the k-th matrix coordinate of M_d^m. Then:

(1) $K[M_d^m]^G$ is generated as a K-algebra by elements of the form

$$s_i(Y_{j_1} \cdots Y_{j_s})$$

for $i \in \{1,\dots,d\}$ and $s \ge 0$, where $Y_k \in \{X_k, X_k^*\}$ and in the orthogonal case $* = \top$ is transposition and in the symplectic case $* = \mathrm{j}$ is symplectic transposition, i.e. $J(-)^\top J^{-1}$ for $J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$. [Zub99, Theorem 1]

(2) The map $K[M_d^m]^G \to K[G^m]^G$ induced by restriction to $G^m \subseteq M_d^m$ is surjective [Zub99, Proposition 3.2]. In particular $K[G^m]^G$ is generated by elements of the form

$$s_i(Y_{j_1} \cdots Y_{j_s})$$

for $i \in \{1,\dots,d\}$ and $s \ge 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

Zubkov proves Theorem 4.1 for algebraically closed fields and then remarks that the claim holds for all infinite fields by a Zariski density argument [Zub99, Remark 3.2].

4.1. Invariants over a field.

We now extend Zubkov's Theorem 4.1 to arbitrary fields and to the groups GSp_{2n}, SO_{2n+1} and GO_n when $n \ge 1$.

Proposition 4.2. Let K be a field and let $m \ge 1$.

(1) Suppose $G \in \{\mathrm{Sp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_n\}$ and $d \in \{2n, 2n+1, n\}$ respectively. Assume further $\mathrm{char}(K) \neq 2$ if $G \in \{\mathrm{SO}_{2n+1}, \mathrm{O}_n\}$. Then $K[G^m]^G$ is generated by elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.
(2) Suppose $G \in \{\mathrm{GSp}_{2n}, \mathrm{GO}_n\}$ and $d \in \{2n, n\}$ respectively. Assume further $\mathrm{char}(K) \neq 2$ if $G = \mathrm{GO}_n$. Then $K[G^m]^G$ is generated by the symplectic (orthogonal) similitude character sim, its inverse sim^{-1} and elements as in (1).

Proof. Let K' be an algebraically closed overfield of K. We first treat the case $G \in \{\mathrm{Sp}_{2n}, \mathrm{O}_n\}$. Let $d = 2n$ in the symplectic case and $d = n$ in the orthogonal case. We have a complex
$$0 \longrightarrow J \longrightarrow K[M_d^m]^G \longrightarrow K[G^m]^G \longrightarrow 0$$
of K-vector spaces, where $J := \ker(K[M_d^m]^G \to K[G^m]^G)$. By faithful flatness it suffices to show, that
$$0 \longrightarrow J \otimes_K K' \longrightarrow K[M_d^m]^G \otimes_K K' \longrightarrow K[G^m]^G \otimes_K K' \longrightarrow 0$$
is exact. By the universal coefficient theorem for rational invariants [Jan03, I.4.18 Proposition], we have isomorphisms $K[M_d^m]^G \otimes_K K' \cong K'[M_d^m]^G$ and $K[G^m]^G \otimes_K K' \cong K'[G^m]^G$, so $J \otimes_K K'$ is the kernel of $K'[M_d^m]^G \to K'[G^m]^G$. The claim follows from Theorem 4.1. For SO_{2n+1}, we note, that the map $K[\mathrm{O}_{2n+1}]^{\mathrm{O}_{2n+1}} \to K[\mathrm{SO}_{2n+1}]^{\mathrm{SO}_{2n+1}}$ is surjective, since $\mathrm{O}_{2n+1} = \mathrm{SO}_{2n+1} \times \{\pm 1\}$.

For the rest of the proof, we argue as in [Wei21, Lemma 3.15]. The natural surjection $\mathrm{Sp}_{2n} \times \mathrm{GL}_1 \to \mathrm{GSp}_{2n}$ induces an inclusion $K[\mathrm{GSp}_{2n}^m]^{\mathrm{GSp}_{2n}} \subseteq K[\mathrm{Sp}_{2n}^m]^{\mathrm{Sp}_{2n}} \otimes_K K[\mathrm{GL}_1^m]$. Here the second factor is generated by the symplectic similitude character sim_i of X_i and its inverse. Since all generators on the right hand side are defined on the left hand side, the map is an isomorphism. For GO_n we argue just the same way. □

4.2. **Invariants over $\mathbb{Z}/p^r$.** In this subsection, fix a prime p and an integer $r \geq 1$. We extend the results over fields to p^r-torsion coefficients by using the theory of good filtrations over $\mathbb{Z}$. We plan to extend the results of this section to general coefficient rings in joint work with Mohamed Moakher. The main purpose here is to demonstrate, that if the coefficients have p^r-torsion the proof is much simpler. We can *lift* invariants using the following variant of Nakayama's lemma.

Lemma 4.3 (Nakayama)**.**

(1) Let M be any $\mathbb{Z}/p^r$-module and assume $M/p = 0$. Then $M = 0$.
(2) Let $f : M \to N$ be a homomorphism of $\mathbb{Z}/p^r$-modules, such that $\overline{f} : M/p \to N/p$ is surjective. Then f is surjective.

Proof. (1) We have $M = pM$, thus $M = p^r M = 0$. (2) We can apply (1) to $\mathrm{coker}(f)$. □

Lemma 4.4. Let G be a Chevalley group and let $S \subseteq \mathbb{Z}[G^m]^G$ be a subset, that generates $\mathbb{F}_p[G^m]^G$ as a ring. Then S generates $\mathbb{Z}/p^r[G^m]^G$.

Proof. Let $A \subseteq \mathbb{Z}/p^r[G^m]^G$ be the subalgebra generated by S. By Proposition 1.11 $\mathbb{Z}[G^m]$ has a good filtration and in particular is acyclic by Lemma 1.9. We calculate
$$\mathbb{Z}/p^r[G^m]^G \otimes_{\mathbb{Z}/p^r} \mathbb{F}_p = (\mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathbb{Z}/p^r) \otimes_{\mathbb{Z}/p^r} \mathbb{F}_p = \mathbb{Z}[G^m]^G \otimes_{\mathbb{Z}} \mathbb{F}_p = \mathbb{F}_p[G^m]^G$$
applying Corollary 1.8 twice. Hence the inclusion induces a surjection $A/p \twoheadrightarrow (\mathbb{Z}/p^r[G^m]^G)/p$. From Lemma 4.3, we obtain $A = \mathbb{Z}/p^r[G^m]^G$. □

Proposition 4.5. Let $\mathcal{O}$ be a commutative ring, such that $p^r\mathcal{O} = 0$. In the following we denote by X_k a generic group element, which can also be understood as a generic matrix under the standard representation. Let $m \geq 1$ and assume $p > 2$ in the orthogonal cases.

(1) Let $n \geq 1$. Then $\mathcal{O}[\mathrm{Sp}_{2n}^m]^{\mathrm{Sp}_{2n}}$ is generated by elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

(2) Let $n \geq 1$. Then $\mathcal{O}[\mathrm{O}_{2n+1}^m]^{\mathrm{SO}_{2n+1}}$ is generated by elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

(3) Let $n \geq 1$. Then $\mathcal{O}[\mathrm{GSp}_{2n}^m]^{\mathrm{GSp}_{2n}}$ is generated by the symplectic similitude character sim, its inverse sim^{-1} and elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

(4) Let $n \geq 1$. Then $\mathcal{O}[\mathrm{GO}_n^m]^{\mathrm{GO}_n}$ is generated by the orthogonal similitude character sim, its inverse sim^{-1} and elements of the form
$$s_i(Y_{j_1} \cdots Y_{j_s})$$
for $i \in \{1, \dots, d\}$ and $s \geq 0$, where $Y_k \in \{X_k, X_k^{-1}\}$.

Proof. Let $G \in \{\mathrm{Sp}_{2n}, \mathrm{O}_{2n+1}, \mathrm{GSp}_{2n}, \mathrm{GO}_n\}$. Since by Proposition 1.11 all $\mathbb{Z}/p^r[(G^0)^m]$ have a good filtration, we may assume $\mathcal{O} = \mathbb{Z}/p^r$. In all cases, the expected generators are defined as elements of $\mathbb{Z}[G^m]^{G^0}$. The claim now follows from Lemma 4.4 and the generators of $\mathbb{F}_p[G^m]^G$ we have given in Proposition 4.2 (Zubkov). □

4.3. **Invariants over $\mathbb{Z}$.** In [DCP17, 15.2.1] de Concini and Procesi have determined the generators of $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n}$ and $\mathbb{Z}[M_n^m]^{\mathrm{SL}_n}$, from which the generators of $\mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ and $\mathbb{Z}[\mathrm{SL}_n^m]^{\mathrm{SL}_n}$ can be deduced. We reprove their result using good filtrations and avoiding usage of the formal character of $\mathbb{Z}[M_n^m]$ and the analysis of root subgroups.

Recall the first fundamental theorem on invariants of several matrices.

Theorem 4.6 (De Concini, Procesi)**.** Let K be an algebraically closed field. Then $K[M_n^m]^{\mathrm{GL}_n}$ is generated by elements of the form
$$s_i(X_{j_1} \cdots X_{j_s})$$
for $i \in \{1, \dots, n\}$ and $s \geq 0$.

Proof. See [DCP17, Theorem 1.10]. □

The first fundamental theorem for SL_n follows right away:

Proposition 4.7. Let K be an algebraically closed field. Then $K[M_n^m]^{\mathrm{SL}_n}$ is generated by elements of the form
$$s_i(X_{j_1} \cdots X_{j_s})$$
for $i \in \{1, \dots, n\}$ and $s \geq 0$.

Proof. The inclusion of the center $\mathrm{GL}_1 \to \mathrm{GL}_n$ and the inclusion $\mathrm{SL}_n \to \mathrm{GL}_n$ combine to a surjection $\mathrm{SL}_n \times \mathrm{GL}_1 \to \mathrm{GL}_n$. Therefore $K[M_n^m]^{\mathrm{GL}_d} = K[M_n^m]^{\mathrm{SL}_n \times \mathrm{GL}_1} = K[M_n^m]^{\mathrm{SL}_n}$ and we conclude by Theorem 4.6. □

To descend to $\mathbb{Z}$, we need the following lemma.

Lemma 4.8. Let $\mathcal{O}$ be a principal ideal domain and let M and M' be finitely generated free $\mathcal{O}$-modules. Let $f : M \to M'$ be an $\mathcal{O}$-module homomorphism, such that for every $\mathcal{O}$-field K the induced map $M \otimes_{\mathcal{O}} K \to M' \otimes_{\mathcal{O}} K$ is an isomorphism. Then f is an isomorphism.

Proof. Taking K the field of fractions of $\mathcal{O}$ shows, that f is injective and that the cokernel C of f is a finitely generated torsion module. For every prime ideal $0 \neq \mathfrak{p} \subseteq \mathcal{O}$, the sequence

$$M \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} \xrightarrow{\sim} M' \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} \longrightarrow C \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} \longrightarrow 0$$

is exact, which shows, that $C \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} = 0$. It follows, that $C = 0$. □

Theorem 4.9 (De Concini, Procesi)**.** $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n} = \mathbb{Z}[M_n^m]^{\mathrm{SL}_n}$ and is generated by elements of the form

$$s_i(X_{j_1} \cdots X_{j_s})$$

for $i \in \{1, \dots, n\}$ and $s \geq 0$.

Proof. $M_n^m = (\mathrm{Std} \otimes \mathrm{Std}^*)^{\oplus m}$, where Std is the standard representation of GL_n. The standard representation of GL_n is self-dual and has a good filtration. We observe, that $\mathbb{Z}[M_n^m]$ admits a grading by finitely generated free $\mathbb{Z}$-modules $S_d := \mathrm{Sym}^d((M_n^m)^*)$, that is preserved by the action of GL_n. The S_d also have a good filtration, so by Lemma 1.9 and Corollary 1.8 $S_d^{\mathrm{GL}_n} \otimes_{\mathbb{Z}} k = (S_d \otimes_{\mathbb{Z}} k)^{\mathrm{GL}_n}$ for any field k.

Let A be a free commutative $\mathbb{Z}$-algebra generated by variables $t_{(i,(j_1,\dots,j_s))}$ with $i \in \{1, \dots, d\}$ and $s \geq 0$. We let $t_{(i,(j_1,\dots,j_s))}$ have degree si and observe that $A = \bigoplus_{d=0}^{\infty} A_d$ is a graded ring, such that each submodule A_d consisting of homogeneous of degree d elements is a finitely generated free $\mathbb{Z}$-module.

The natural map $A \to \mathbb{Z}[M_n^m]^{\mathrm{GL}_n}$ sending $t_{(i,(j_1,\dots,j_s))}$ to $s_i(X_{j_s} \dots X_{j_s})$ is graded. By Theorem 4.6, the maps $A_d \otimes_{\mathbb{Z}} k \to S_d^{\mathrm{GL}_n} \otimes_{\mathbb{Z}} k$ are surjective for every algebraically closed field k. Hence by Lemma 4.8 the maps $A_d \to S_d$ are surjective and thus $A \to S$ is surjective, proving the first claim. The argument for the SL_n-invariants is the same, using Proposition 4.7. □

To pass from invariants of $\mathbb{Z}[M_n^m]$ to invariants of $\mathbb{Z}[\mathrm{GL}_n^m]$ and $\mathbb{Z}[\mathrm{SL}_n^m]$, we use the following general lemma.

Lemma 4.10. Let G be a split reductive group over $\mathbb{Z}$ and let

$$0 \to C \to B \to A \to 0$$

$$0 \to C' \to B' \to A' \to 0$$

be two short exact sequences of G-modules with good filtration. Then the map $(B \otimes B')^G \to (A \otimes A')^G$ is surjective.

Proof. Since good filtration modules are free, the sequences

$$0 \to C \otimes A' \to B \otimes A' \to A \otimes A' \to 0$$

$$0 \to B \otimes C' \to B \otimes B' \to B \otimes A' \to 0$$

are exact. By Mathieu's tensor product theorem Theorem 1.10, the modules $C \otimes A'$ and $B \otimes C'$ have good filtrations, hence by Lemma 1.9 the maps $(B \otimes A')^G \to (A \otimes A')^G$ and $(B \otimes B')^G \to (B \otimes A')^G$ are surjective. □

Theorem 4.11. Let $\mathcal{O}$ be a commutative ring, let $m \geq 1$ and let $n \geq 1$.

(1) $\mathcal{O}[M_n^m]^{\mathrm{GL}_n}$ and $\mathcal{O}[M_n^m]^{\mathrm{SL}_n}$ are generated by elements of the form

$$s_i(X_{j_1} \cdots X_{j_s})$$

for $i \in \{1, \dots, n\}$ and $s \geq 0$.

(2) $\mathcal{O}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ and $\mathcal{O}[\mathrm{SL}_n^m]^{\mathrm{SL}_n}$ are generated by elements of the form

$$s_i(X_{j_1} \cdots X_{j_s})$$

for $i \in \{1, \dots, n\}$ and $s \geq 0$ and $\det^{-1}(X_j)$ for $j \in \{1, \dots, m\}$.

Proof. By Proposition 1.11, it is for both GL_n and SL_n sufficient to prove the claim for $\mathcal{O} = \mathbb{Z}$. The closed immersion $\mathrm{GL}_n \to M_n \times \mathbb{A}^1,\ g \mapsto (g, \det(g)^{-1})$ induces a surjection $\mathbb{Z}[(M_n \times \mathbb{A}^1)^m] \to \mathbb{Z}[\mathrm{GL}_n^m]$ of $\mathbb{Z}$-graded $\mathbb{Z}$-modules with GL_n-action, where the graded pieces are finitely generated free $\mathbb{Z}$-modules.

Identifying $\mathbb{Z}[\mathbb{A}^1] = \mathbb{Z}[t]$, we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}[M_n] \otimes \mathbb{Z}[t] \xrightarrow{\cdot(t \cdot \det -1)} \mathbb{Z}[M_n] \otimes \mathbb{Z}[t] \longrightarrow \mathbb{Z}[\mathrm{GL}_n] \longrightarrow 0$$

of GL_d-modules, since $t \cdot \det -1$ is an invariant element of the integral domain $\mathbb{Z}[M_d] \otimes \mathbb{Z}[t]$. By Lemma 4.10 and since $\mathbb{Z}[\mathrm{GL}_d]$ and $\mathbb{Z}[M_d]$ have good filtrations (Proposition 1.11, Proposition 1.12), the maps $\mathbb{Z}[M_n^m]^{\mathrm{GL}_n} \otimes \mathbb{Z}[t]^{\otimes m} = \mathbb{Z}[(M_n \times \mathbb{A}^1)^m]^{\mathrm{GL}_n} \to \mathbb{Z}[\mathrm{GL}_n^m]^{\mathrm{GL}_n}$ are surjective. The claim follows from Theorem 4.9.

The same argument using the closed immersion $\mathrm{SL}_n \to M_n$ and the short exact sequence

$$0 \longrightarrow \mathbb{Z}[M_n] \xrightarrow{\cdot(\det -1)} \mathbb{Z}[M_n] \longrightarrow \mathbb{Z}[\mathrm{SL}_n] \longrightarrow 0$$

implies the claim on $\mathbb{Z}[\mathrm{SL}_n^m]^{\mathrm{SL}_n}$. □

In upcoming joint work with Mohamed Moakher, we will compute explicit generators of $\mathbb{Z}[G^m]^G$ for $G \in \{\mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$ building on work of Zubkov.

Corollary 4.12. Let $\mathcal{O}$ be a commutative ring.

(1) The $\mathcal{F}$-$\mathcal{O}$-algebra $\mathcal{O}[\mathrm{GL}_n^\bullet]^{\mathrm{GL}_n}$ is generated by $s_1, \dots, s_n \in \mathcal{O}[\mathrm{GL}_n]^{\mathrm{GL}_n}$.
(2) The $\mathcal{F}$-$\mathcal{O}$-algebra $\mathcal{O}[\mathrm{SL}_n^\bullet]^{\mathrm{SL}_n}$ is generated by $s_1, \dots, s_{n-1} \in \mathcal{O}[\mathrm{SL}_n]^{\mathrm{SL}_n}$.

Proof. This follows by inspection of the generators computed in Theorem 4.11 and substitutions. □

Corollary 4.13. Let $r \geq 1$ be an integer and let $\mathcal{O}$ be a commutative ring, such that $p^r\mathcal{O} = 0$. Let $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_{2n+1}\}$ and assume that $p > 2$ in the orthogonal cases. Then the maps $\mathcal{O}[\mathrm{GL}_n^m]^{\mathrm{GL}_n} \to \mathcal{O}[G^m]^{G^0}$ are surjective for all $m \geq 1$. In particular the $\mathcal{F}$-$\mathcal{O}$-algebras $\mathcal{O}[G^\bullet]^{G^0}$ are finitely generated.

Proof. This follows from Corollary 4.12 and Proposition 4.5 and substitutions. □

5. Deformations of G-pseudocharacters

5.1. **Coefficient rings.** Fix a prime $p > 0$. Let κ be a field of one of the following three types:

(1) κ is a finite discrete field.
(2) κ is a finite extension of $\mathbb{Q}_p$ equipped with the p-adic topology.
(3) κ is a finite extension of $\mathbb{F}_p((t))$ equipped with the t-adic topology.

We introduce a coefficient ring Λ in each of the three cases for κ.

(1) In case (1), let Λ be the ring of integers of a p-adic field with residue field κ.
(2) In case (2), let $\Lambda = \kappa$.
(3) In case (3), let $\Lambda = \kappa$.

By slight abuse of terminology, we will call only such rings Λ *coefficient rings* for κ.

Let $\mathfrak{A}_\Lambda$ be the category of artinian local Λ-algebras with residue field κ. Every A in $\mathfrak{A}_\Lambda$ has a canonical projection $\pi_A : A \to \kappa$ with kernel $\mathfrak{m}_A$ the maximal ideal of A. Note, that $\mathfrak{A}_\Lambda$ admits fiber products [Til96, §2.2]. Every complete local Λ-algebra A with residue field κ is algebraically isomorphic to the inverse limit $\varprojlim A/\mathfrak{m}_A^n$. If A is complete local noetherian Λ-algebra, it can be written as $A = \Lambda[[X_1, \dots, X_r]]/I$, where r is the κ-dimension of the relative cotangent space $t_A^* = \mathfrak{m}_A/(\mathfrak{m}_A^2 + \mathfrak{m}_\Lambda A)$ of A [Til96, Lem. 5.1].

5.2. **The universal deformation ring $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$.**

Definition 5.1. Let κ be a finite or a local field and let Λ be a coefficient ring as in Section 5.1 with residue field κ. Let Γ be a profinite group and let G be a generalized reductive Λ-group scheme. Let $\overline{\Theta} \in \mathrm{cPC}^\Gamma_G(\kappa)$ be a continuous G-pseudocharacter of Γ. We define the *deformation functor* of $\overline{\Theta}$

$$\begin{aligned} \mathrm{Def}_{\overline{\Theta}} : \mathfrak{A}_\Lambda &\to \mathrm{Set} \\ A &\mapsto \{\Theta \in \mathrm{cPC}^\Gamma_G(A) \mid \Theta \otimes_A \kappa = \overline{\Theta}\} \end{aligned}$$

that sends an object $A \in \mathfrak{A}_\Lambda$ to the set of continuous G-pseudocharacters Θ of Γ over A with $\Theta \otimes_A \kappa = \overline{\Theta}$.

If A is an arbitrary local topological Λ-algebra with residue field κ, we define $\mathrm{Def}_{\overline{\Theta}}(A)$ analogously. This is notation for a single A and shall not extend the deformation functor $\mathrm{Def}_{\overline{\Theta}}$. To prove pro-representability of the deformation functor we need to show, that it is compatible with taking inverse limits.

Lemma 5.2. Let Λ be a coefficient ring as in Section 5.1 and let $A = \varprojlim_i A_i$ be a projective limit of local topological Λ-algebras with $A_i \in \mathfrak{A}_\Lambda$, endowed with the projective limit topology. Let $\overline{\Theta} \in \mathrm{cPC}^\Gamma_G(\kappa)$. Then the natural map $\mathrm{Def}_{\overline{\Theta}}(A) \to \varprojlim_i \mathrm{Def}_{\overline{\Theta}}(A_i)$ is bijective.

Proof. Per definition, we have a pullback diagram

$$\begin{array}{ccc} \mathrm{Def}_{\overline{\Theta}}(A) & \longrightarrow & \mathrm{cPC}^\Gamma_G(A) \\ \downarrow & & \downarrow \\ \{\overline{\Theta}\} & \longrightarrow & \mathrm{cPC}^\Gamma_G(\kappa) \end{array}$$

So it suffices to prove the claim for cPC^Γ_G instead of the deformation functor.

By Corollary 3.10 and since $\mathcal{C}(\Gamma^n, A) = \varprojlim_i \mathcal{C}(\Gamma^n, A_i)$, we have

$$\begin{aligned} \mathrm{cPC}_G^\Gamma(A) &= \mathrm{Hom}_{\mathrm{CAlg}_\Lambda^{\mathcal{T}}}(\Lambda[G^\bullet]^{G^0},\ \mathcal{C}(\Gamma^\bullet, A)) \\ &= \mathrm{Hom}_{\mathrm{CAlg}_\Lambda^{\mathcal{T}}}(\Lambda[G^\bullet]^{G^0},\ \varprojlim_i \mathcal{C}(\Gamma^\bullet, A_i)) \\ &= \varprojlim_i \mathrm{Hom}_{\mathrm{CAlg}_\Lambda^{\mathcal{T}}}(\Lambda[G^\bullet]^{G^0},\ \mathcal{C}(\Gamma^\bullet, A_i)) \\ &= \varprojlim_i \mathrm{cPC}_G^\Gamma(A_i) \end{aligned}$$

□

Theorem 5.3. Let κ be a finite or a local field and let Λ be a coefficient ring for κ. Let Γ be a profinite group and let G be a generalized reductive Λ-group scheme. Let $\overline{\Theta} \in \mathrm{cPC}_G^\Gamma(\kappa)$ be a continuous pseudocharacter. Then the deformation functor

$$\mathrm{Def}_{\overline{\Theta}} : \mathfrak{A}_\Lambda \to \mathrm{Set}$$

is pro-representable by some inverse limit $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ of artinian Λ-algebras with residue field κ, endowed with the inverse limit topology. If κ is finite, then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is pro-p and in particular complete.

If $\overline{\Theta}$ is induced by a continuous representation $\overline{\rho} : \Gamma \to G(\kappa)$, we write $R^{\mathrm{ps}}_{\Lambda,\overline{\rho}} := R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$. If the residue field is a local field, we only have one choice for Λ and we will usually drop it from notations.

Proof. We adapt the proof of [Che14, Proposition 3.3]. Let $B := B_G^\Gamma$ be the Λ-algebra from Theorem 3.11, that represents $\mathrm{PC}_G^\Gamma : \mathrm{CAlg}_\Lambda \to \mathrm{Set}$. Let $\Theta^u \in \mathrm{PC}_G^\Gamma(B)$ be the universal G-pseudocharacter and $\psi : B \to \kappa$ the morphism, that corresponds to $\overline{\Theta}$ under the identification $\mathrm{Hom}_{\mathrm{CAlg}_\Lambda}(B, \kappa) \cong \mathrm{PC}_G^\Gamma(\kappa)$. We define a set $\mathcal{I}$ of ideals of B as follows: An ideal $I \subseteq B$ is in $\mathcal{I}$, if and only if the following three conditions hold:

(1) I is contained in the maximal ideal $\mathfrak{m} := \ker(\psi)$ associated with ψ.
(2) B/I is artinian and local. If κ is finite, we equip B/I with the discrete topology. If κ is a local field then B/I is a finite-dimensional κ-vector space and we equip B/I with the product topology of κ.
(3) The image $\Theta^I := \pi^I_* \Theta^u$ of Θ^u under the map $\mathrm{PC}_G^\Gamma(B) \to \mathrm{PC}_G^\Gamma(B/I)$ induced by the projection $\pi^I : B \to B/I$ is a continuous G-pseudocharacter.

$(\mathcal{I}, \subseteq)$ is a cofiltered poset: If $I, J \in \mathcal{I}$, then we have

(1) $I \cap J \subseteq \mathfrak{m}$.
(2) The map $\iota : B/(I \cap J) \to B/I \times B/J$ is injective, hence $B/(I \cap J)$ is artinian. Let $\mathfrak{m}'$ be a maximal ideal of B, that contains $I \cap J$. Then $IJ \subset \mathfrak{m}'$, hence either $I \subset \mathfrak{m}'$ or $J \subset \mathfrak{m}'$. In both cases $\mathfrak{m}' = \mathfrak{m}$, since B/I and B/J are local. Hence $B/(I \cap J)$ is local.
(3) Note, that ι is a topological embedding. Thus, for the reduction $\Theta^{I \cap J}$ of Θ^u mod $I \cap J$ the homomorphism $\Theta_n^{I \cap J} : B[G^n]^{G^0} \to \mathrm{Map}(\Gamma^n, B/(I \cap J))$ has image in $\mathcal{C}(\Gamma^n, B/(I \cap J))$ for all $n \geq 1$.

Define the topological Λ-algebra

$$R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} := \varprojlim_{I \in \mathcal{I}} B/I$$

The inverse limit is taken in the category of topological Λ-algebras. Let $\pi_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}} : R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to \kappa$ be the map induced by the identification $B/\ker(\psi) \cong \kappa$ and let $\mathfrak{m}_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}} := \ker(\pi_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}})$. Each B/I is a local ring with residue field κ, so an element of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is invertible if and only if its reduction to κ is. This shows, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is local with maximal ideal $\mathfrak{m}_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}}$.

If κ is finite, then each B/I is a finite p-group and $R^{\mathrm{ps}}_{\overline{\Theta}}$ is pro-p and in particular complete.

We show, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ pro-represents $\mathrm{Def}_{\overline{\Theta}}$ and that $\iota_*\Theta^u \in \mathrm{Def}_{\overline{\Theta}}(R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}})$ is the universal deformation of $\overline{\Theta}$, where $\iota : B \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is the canonical map. Assume for the proof, that $\mathrm{Def}_{\overline{\Theta}}$ is defined on the category of local topological Λ-algebras with residue field κ. This way we get uniqueness of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ once we show representability. By Lemma 5.2 we have an isomorphism

$$\mathrm{Def}_{\overline{\Theta}}(R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}) \cong \varprojlim_{I \in \mathcal{I}} \mathrm{Def}_{\overline{\Theta}}(B/I)$$

so it suffices to show representability for artinian rings.

If $A \in \mathfrak{A}_\Lambda$ and $\Theta \in \mathrm{Def}_{\overline{\Theta}}(A)$, then Θ corresponds to a unique homomorphism $\phi : B \to A$, such that $\phi_*\Theta^u = \Theta$ and $\phi \mod \mathfrak{m}_A = \psi$. We will show, that $\ker(\phi) \in \mathcal{I}$. We have $\ker(\phi) \subset \ker(\psi) = \mathfrak{m}$ and $B/\ker(\phi) \subseteq A$ is artinian local. We have to show, that $\pi^{\ker(\phi)}_*\Theta^u$ is continuous. Indeed $\overline{\phi}_*\pi^{\ker(\phi)}_*\Theta^u = \phi_*\Theta^u = \Theta$ is continuous, where $\overline{\phi} : B/\ker(\phi) \to A$ is the map induced by ϕ. Since $\overline{\phi}$ is a topological embedding $\pi^{\ker(\phi)}_*\Theta^u$ is continuous. So there is a unique factorization $B \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to B/\ker(\phi) \to A$ of ϕ over a continuous map $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to A$.

For the converse suppose, that $\varphi : R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to A$ is a continuous local Λ-homomorphism compatible with the projections to κ. We have to show, that the pseudocharacter $\varphi_*\iota_*\Theta^u$ is continuous. It is enough to show, that the universal deformation $\iota_*\Theta^u$ is continuous. Let $\tilde{\pi}^I : R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to B/I$ for $I \in \mathcal{I}$ be the projection map from the definition of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ as an inverse limit. The pseudocharacters $\tilde{\pi}^I_*\iota_*\Theta^u = \pi^I_*\Theta^u$ are continuous by definition of $\mathcal{I}$. For fixed $m \geq 1$ and $f \in \Lambda[G^m]^{G^0}$ the map $(\iota_*\Theta^u)_m(f) : \Gamma^m \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ will be continuous by the universal property of limits. □

Now that we have proved existence of universal pseudodeformation rings, we observe, that certain completed local rings at dimension 1 points x are pseudodeformation rings for a deformation problem with residue field $\kappa(x)$. It is for this reason, that we also treat cases (2) and (3) from the beginning of this section.

Proposition 5.4. Let Γ be a profinite group. Let κ be a finite field and let Λ be a coefficient ring for κ. Let $\overline{\Theta} \in \mathrm{cPC}^\Gamma_G(\kappa)$ and let $x \in \mathrm{Spec}(R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}})$ be a dimension 1 point and residue field $\kappa(x)$. Assume, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian. By [BIP21, Lemma 3.16] $\kappa(x)$ is a local field and the induced pseudocharacter $\Theta_x \in \mathrm{cPC}^\Gamma_G(\kappa(x))$ is continuous. Let $\mathfrak{p} := \ker(\kappa(x) \otimes_\Lambda R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to \kappa(x))$. Then the following two rings are canonically isomorphic:

(1) The universal pseudodeformation ring $R^{\mathrm{ps}}_{\Theta_x}$.
(2) The completion of $\kappa(x) \otimes_\Lambda R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ at $\mathfrak{p}$.

The isomorphism is given by the induced map $\kappa(x) \otimes_\Lambda R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \to R^{\mathrm{ps}}_{\Theta_x}$.

Proof. The proof of [BJ19, Corollary 4.8.7] goes through in our setting. □

5.3. **Noetherianity for topologically finitely generated profinite groups.**

Lemma 5.5. Let Γ be a topological group, $\Delta \subseteq \Gamma$ a dense subgroup, $\mathcal{O}$ a commutative ring and G a generalized reductive $\mathcal{O}$-group scheme. Then for all Hausdorff $\mathcal{O}$-algebras A the restriction

$$\mathrm{cPC}_G^\Gamma(A) \to \mathrm{cPC}_G^\Delta(A)$$

defined by composition with $\mathcal{C}(\Gamma^n, A) \to \mathcal{C}(\Delta^n, A)$ is injective.

This is a generalization of the density argument in [Che14, Ex. 2.31].

Proof. Let $\Theta, \Theta' \in \mathrm{cPC}_G^\Gamma(A)$ be such that $\Theta|_\Delta = \Theta'|_\Delta$. Let $n \geq 0$ and $f \in \mathcal{O}[G^n]^{G^0}$. Then $\Theta_n(f), \Theta'_n(f) : \Gamma^n \to A$ are continuous maps, that agree on the dense subset $\Delta^n \subseteq \Gamma^n$, hence must be equal. □

Lemma 5.6. Let Γ be a group, let G and H be generalized reductive group schemes over a commutative ring $\mathcal{O}$ and let $\iota : G \to H$ be a homomorphism of $\mathcal{O}$-group schemes, such that the induced map of $\mathcal{F}$-$\mathcal{O}$-algebras $\mathcal{O}[H^\bullet]^{H^0} \to \mathcal{O}[G^\bullet]^{G^0}$ is surjective. Let A be a commutative $\mathcal{O}$-algebra and let $\Theta \in \mathrm{PC}_G^\Gamma(A)$. Then $\ker(\Theta) = \ker(\iota(\Theta))$.

Proof. By inspection of the Definition 2.5 of kernel. □

Examples, that satisfy the hypotheses of Lemma 5.6 can be obtained from Corollary 4.13.

Proposition 5.7. Let Λ be the ring of integers of a p-adic local field with residue field κ. Let A be a pro-p local Λ-algebra with residue field κ. The following are equivalent:

(1) A is noetherian.
(2) $\mathfrak{m}_A$ is a finitely generated ideal.
(3) $\mathfrak{m}_A/\mathfrak{m}_A^2$ is a finite-dimensional κ-vector space.
(4) $\mathfrak{m}_A/(\mathfrak{m}_\Lambda + \mathfrak{m}_A^2)$ is a finite-dimensional κ-vector space.

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4$ is clear. The proof of $4 \Rightarrow 1$ can be found in Hida's notes [Hid14, Lemma 2.10]. □

Proposition 5.8. Assume, that Λ is the ring of integers of a p-adic local field with residue field κ. Let Γ be a group, let G be a generalized reductive Λ-group scheme and let $\overline{\Theta} \in \mathrm{cPC}_G^\Gamma(\kappa)$. Then the following are equivalent:

(1) $\dim_\kappa(\mathrm{Def}_{\overline{\Theta}}(\kappa[\varepsilon])) < \infty$.
(2) $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is a noetherian ring.

Proof. Since $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ represents $\mathrm{Def}_{\overline{\Theta}}$ (Theorem 5.3), the relative tangent space $(\mathfrak{m}_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}}/(\mathfrak{m}_\Lambda + \mathfrak{m}^2_{R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}}))^*$ of $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ over Λ identifies with $\mathrm{Def}_{\overline{\Theta}}(\kappa[\varepsilon])$. Since $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is pro-p, the claim follows from Proposition 5.7. □

Theorem 5.9. Assume, that Λ is the ring of integers of a p-adic local field with residue field κ and that G is a generalized reductive Λ-group scheme. Let Γ be a topologically finitely generated profinite group and let $\overline{\Theta} \in \mathrm{cPC}_G^\Gamma(\kappa)$. Then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian.

Proof. Let $\Delta \subseteq \Gamma$ be a dense and finitely generated subgroup of Γ. We have a sequence of injections

$$\mathrm{Def}_{\Lambda,\overline{\Theta}}(\kappa[\varepsilon]) \subseteq \mathrm{cPC}_G^\Gamma(\kappa[\varepsilon]) \overset{5.5}{\subseteq} \mathrm{cPC}_G^\Delta(\kappa[\varepsilon]) \subseteq \mathrm{PC}_G^\Delta(\kappa[\varepsilon]) \overset{3.11}{\cong} \mathrm{Hom}_\Lambda(B_G^\Delta, \kappa[\varepsilon])$$

By [Sta22, 032W] and [Sta22, 0334] Λ is universally Japanese. By Proposition 3.12, $\mathrm{Hom}_\Lambda(B_G^\Delta, \kappa[\varepsilon])$ is a finite-dimensional κ-vector space. By Proposition 5.8 we conclude, that $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian. □

5.4. **Noetherianity for profinite groups satisfying Φ_p.** The idea of establishing noetherianity of the pseudodeformation rings $R^{\mathrm{ps}}_{\overline{\Theta}}$ for a classical group G in case we only know that our profinite group Γ satisfies Mazur's condition Φ_p is to prove surjectivity of the transition map $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\overline{\Theta}}$ for a suitable rational representation $\iota : G \to \mathrm{GL}_n$, and use noetherianity of $R^{\mathrm{ps}}_{\iota(\overline{\Theta})}$. In this section we give a criterion in terms of invariant theory, which can be applied to other reductive groups once their invariant theory is understood. We found the proofs of this section before the argument in Proposition 3.12 was found, which is of course general and sufficient for applications to absolute Galois groups of local fields.

Lemma 5.10. Let Γ be a group and let $\iota : G \to G'$ be a homomorphism of generalized reductive group schemes over a commutative ring $\mathcal{O}$. Suppose, that the map $\mathcal{O}[G'^{\bullet}]^{G'^0} \to \mathcal{O}[G^{\bullet}]^{G^0}$ of $\mathcal{F}$-$\mathcal{O}$-algebras is surjective. Then the map $\iota^* : B^{\Gamma}_{G'} \to B^{\Gamma}_{G}$ induced by ι is surjective.

Proof. By Theorem 3.11 it is enough to show, that for each $m \geq 1$, each $\mu \in \mathcal{O}[G^m]^{G^0}$ and each $\gamma = (\gamma_1, \dots, \gamma_m) \in \Gamma^m$, the element $t_{\mu,\gamma} \in B^{\Gamma}_G$ has a preimage in $B^{\Gamma}_{G'}$. By surjectivity of $\mathcal{O}[G'^m]^{G'^0} \to \mathcal{O}[G^m]^{G^0}$, we find some $\mu' \in \mathcal{O}[G'^m]^{G'^0}$ mapping to μ. We claim, that $\iota^*(t_{\mu',\gamma}) = t_{\mu,\gamma}$. Let $A \in \mathrm{CAlg}_{\mathcal{O}}$, $\Theta \in \mathrm{PC}^{\Gamma}_G(A)$ and $f_\Theta : B^{\Gamma}_G \to A$ the homomorphism attached to Θ. Let $f_{\iota(\Theta)} : B^{\Gamma}_{G'} \to A$ be the homomorphism attached to $\iota(\Theta)$. By definition $f_\Theta(\iota^*(t_{\mu',\gamma})) = f_{\iota(\Theta)}(t_{\mu',\gamma}) = \iota(\Theta)_m(\mu')(\gamma) = \Theta_m(\gamma)$. Since this characterizes $\iota^*(t_{\mu',\gamma})$ uniquely, we have $\iota^*(t_{\mu',\gamma}) = t_{\mu,\gamma}$. □

Lemma 5.11. Let Γ be a profinite group. Let G and G' be generalized reductive group schemes over a coefficient ring Λ with finite residue field κ. Let $\iota : G \to G'$ be a homomorphism of Λ-group schemes. Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$ be a continuous pseudocharacter and we denote by $\iota(\overline{\Theta})$ its image in $\mathrm{cPC}^{\Gamma}_{G'}(\kappa)$. Assume, that the homomorphism $B^{\Gamma}_{G'}/p \to B^{\Gamma}_G/p$ is surjective. Then the natural homomorphism $R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is surjective.

Proof. Let $B := B^{\Gamma}_G/p$, $B' := B^{\Gamma}_{G'}/p$, $R := R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}/p$, $R' := R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})}/p$ and let $J := \ker(B' \to B)$. By Nakayama's lemma it is enough to show, that the natural map $j : R' = \varprojlim_{I' \in \mathcal{I}'} B'/I' \to \varprojlim_{I \in \mathcal{I}} B/I = R$ induced by ι is surjective. Here the ideals $\mathcal{I}$ and $\mathcal{I}'$ are defined as in the proof of Theorem 5.3. To an ideal $I \in \mathcal{I}$, we attach the ideal $j^{-1}(I)$ and we claim, that $j^{-1}(I) \in \mathcal{I}'$ and this induces a well-defined map of cofiltered sets $j^{-1} : \mathcal{I} \to \mathcal{I}'$.

As in the proof of Theorem 5.3, let $\psi : B \to \kappa$ be the homomorphism attached to $\overline{\Theta}$. Then $\psi' := \psi \circ j$ is the homomorphism attached to $\iota(\overline{\Theta})$. Let $\mathfrak{m} := \ker(\psi)$ and $\mathfrak{m}' := \ker(\psi')$. We observe, that $\mathfrak{m}' = j^{-1}(\mathfrak{m})$ and thus $j^{-1}(I) \subseteq \mathfrak{m}'$ for all $I \in \mathcal{I}$. For the second property in the definition of $\mathcal{I}'$, we observe, that $B'/j^{-1}(I) \to B/I$ is injective, and surjectivity follows as $pB = 0$. So $B'/j^{-1}(I) \cong B/I$ is finite. Let $\Theta^u \in \mathrm{PC}^{\Gamma}_G(B)$ and $\Theta^{u'} \in \mathrm{PC}^{\Gamma}_{G'}(B')$ be the universal pseudocharacters mod p. The pseudocharacter $\pi^{j^{-1}(I)}_* \Theta^{u'} = \iota(\pi^I_* \Theta^u)$ is continuous as the image of a continuous pseudocharacter.

Next, we claim, that the map $\mathcal{I}' \to \mathcal{I}$, $I' \mapsto j(I' + J)$ is surjective. Indeed, if $I \in \mathcal{I}$, we have just shown, that $j^{-1}(I) \in \mathcal{I}'$ and $j(j^{-1}(I) + J) = j(j^{-1}(I)) = I$. We therefore obtain an isomorphism $R \cong \varprojlim_{I' \in \mathcal{I}'} B'/(I' + J)$ and the map between deformation rings is now a naturally induced map between limits over $\mathcal{I}'$.

The image T of R' in R is compact, since R' is profinite. It is dense, since for all $I' \in \mathcal{I}'$, the map $B'/I' \to B'/(I' + J)$ is surjective. As an inverse limit of Hausdorff spaces R is Hausdorff and hence T is closed in R. It follows, that $T = R$. □

Theorem 5.12. Let $G \in \{\mathrm{SL}_n, \mathrm{GL}_n, \mathrm{Sp}_{2n}, \mathrm{GSp}_{2n}, \mathrm{SO}_{2n+1}, \mathrm{O}_{2n+1}, \mathrm{GO}_n\}$ over a coefficient ring Λ with finite residue field κ and assume $p > 2$ in the orthogonal cases. Let $\iota : G \to \mathrm{GL}_d$ be the standard representation of G. Let Γ be a profinite group and let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_{G}(\kappa)$ be a continuous pseudocharacter. Then the canonical map $R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is surjective. If in addition Γ satisfies Mazur's condition Φ_p, then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is noetherian.

Proof. We have shown in Corollary 4.13, that for $m \geq 1$ the natural maps $\mathcal{O}/p[\mathrm{GL}_d^m]^{\mathrm{GL}_n} \to \mathcal{O}/p[G^m]^{G^0}$ are surjective. It follows from Lemma 5.10, that the maps $B^{\Gamma}_{\mathrm{GL}_{d,\Lambda/p}} \to B^{\Gamma}_{G_{\Lambda/p}}$ are surjective. By Proposition 3.13, we have surjections $B^{\Gamma}_{\mathrm{GL}_d}/p \to B^{\Gamma}_{G}/p$. Hence we can apply Lemma 5.11 and see, that the map $R^{\mathrm{ps}}_{\Lambda,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is surjective. □

6. Comparison with Chenevier's construction

6.1. **Determinant laws.** Chenevier's definition of pseudocharacters relies on the idea of a generalized determinant. He uses the language of polynomial laws. We recall the basic definitions.

Let A be a commutative ring and M and N be two A-modules. The association $B \mapsto M \otimes_A B$ defines a functor $\underline{M} : \mathrm{CAlg}_A \to \mathrm{Set}$ on the category of commutative A-algebras CAlg_A. Any A-linear map $f : M \to N$ gives rise to a natural transformation $\underline{M} \to \underline{N}$.

Definition 6.1. Let A be a commutative ring and M and N be two A-modules.

(i) An *A-polynomial law* $P : M \to N$ is a natural transformation of functors $f : \underline{M} \to \underline{N}$.
(ii) An A-polynomial law $P : M \to N$ is *homogeneous* of degree $n \geq 0$, if $P_B(bx) = b^n P_B(x)$ for all $B \in \mathrm{CAlg}_A$, all $b \in B$ and all $x \in M \otimes_A B$.
(iii) When $M = R$ is a unital (not necessarily commutative) associative A-algebra, then $P : R \to N$ is *multiplicative*, if $P_B(1) = 1$ and $P_B(xy) = P_B(x)P_B(y)$ for all $B \in \mathrm{CAlg}_A$ and all $x, y \in R \otimes_A B$.

Definition 6.2. Let A be a commutative ring, R be any A-algebra and $d \geq 1$ an integer. A *d-dimensional A-valued determinant law* is a multiplicative A-polynomial law $D : R \to A$, that is homogeneous of degree d.

In this text, we are going to consider determinants laws, only when $R = A[\Gamma]$ is a group algebra. We denote the set of A-valued determinant laws $D : A[\Gamma] \to A$ by $\mathrm{Det}^{\Gamma}_d(A)$.

If $\rho : \Gamma \to \mathrm{GL}_d(A)$ is a representation, then we obtain a determinant law $D_\rho \in \mathrm{Det}^{\Gamma}_d(A)$ by first extending ρ to a homomorphism of A-algebras $\rho : A[\Gamma] \to M_d(A)$ and the setting $D_{\rho,B}(r) := \det(\rho(r))$ for all $r \in B[\Gamma]$. This defines a map $\mathrm{Rep}^{\Gamma,\square}_{\mathrm{GL}_d}(A) \to \mathrm{Det}^{\Gamma}_d(A)$, which is natural in A and Γ.

Definition 6.3. Let D be an A-linear d-dimensional determinant law. We define the coefficients $\Lambda_i : R \to A$ of the characteristic polynomial of D by the expansion

$$\chi^D(r,t) = D_{B[t]}(t-r) = \sum_{i=0}^{d} (-1)^i \Lambda_{i,B}(r) t^{d-i} \in B[t]$$

for all $B \in \mathrm{CAlg}_A$.

One can show, that the coefficients Λ_i give rise to i-homogeneous A-polynomial laws.

6.2. **Emerson's bijection.** Kathleen Emerson has proven in her 2018 dissertation [Eme18], that there is a bijection between GL_d-valued pseudocharacters and d-dimensional determinant laws over any base ring. We summarize her results and adapt them to the continuous case in Section 6.3. In this section we consider GL_d as a group scheme over $\mathbb{Z}$.

Theorem 6.4. Let A be a commutative ring, Γ a group and $d \geq 1$. Then the map

$$\mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A) \to \mathrm{Det}^{\Gamma}_d(A), \quad \Theta \mapsto D_\Theta$$

defined in [Eme18, Theorem 4.0.1] is a well-defined bijection.

Emerson's bijection is characterized by the following property: If s_i for $1 \leq i \leq d$ are the coefficients of the characteristic polynomial of a generic matrix in GL_d viewed as elements of $\mathbb{Z}[\mathrm{GL}_d]^{\mathrm{GL}_d}$, then a GL_d-pseudocharacter $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A)$ corresponds to a d-dimensional determinant law $D \in \mathrm{Det}^{\Gamma}_d(A)$ if and only if $\Lambda_{i,A}(\gamma) = \Theta_1(s_i)(\gamma)$ for all $\gamma \in \Gamma$.

In particular if $\rho : \Gamma \to \mathrm{GL}_d(A)$ is a representation, then $D_{\Theta_\rho} = D_\rho$.

6.3. **Continuous determinant laws.** Let Γ be a topological group and let A be a topological ring. We say, that a d-dimensional A-linear determinant law $D \in \mathrm{Det}^{\Gamma}_d(A)$ is *continuous*, if the coefficients Λ_i of the characteristic polynomial of D give rise to continuous maps $\Lambda_{i,A}|_\Gamma : \Gamma \to A$. This notion of continuity is equivalent to that defined in [Che14, §2.30]. We denote the set of continuous d-dimensional A-linear determinant laws by $\mathrm{cDet}^{\Gamma}_d(A)$.

If $\rho : \Gamma \to \mathrm{GL}_d(A)$ is a continuous representation, then D_ρ is a continuous determinant law. So we have a map $\mathrm{cRep}^{\Gamma,\square}_{\mathrm{GL}_d}(A) \to \mathrm{cDet}^{\Gamma}_d(A)$, which is natural in A and Γ.

Proposition 6.5. Let A be a commutative topological ring, Γ a topological group and $d \geq 1$. Then $\Theta \in \mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A)$ is continuous if and only if D_Θ is continuous. In particular the bijection $\mathrm{PC}^{\Gamma}_{\mathrm{GL}_d}(A) \to \mathrm{Det}^{\Gamma}_d(A)$, $\Theta \mapsto D_\Theta$ in Theorem 6.4 restricts to a bijection $\mathrm{cPC}^{\Gamma}_{\mathrm{GL}_d}(A) \to \mathrm{cDet}^{\Gamma}_d(A)$.

Proof. First suppose, that Θ is continuous. Then $\Lambda_{i,A}|_\Gamma = \Theta_1(s_i)$ is a continuous map by definition of continuity of Θ, hence D_Θ is continuous.

Conversely, if D_Θ is continuous, then $\Theta_1(s_i)$ is continuous for all $1 \leq i \leq d$. Since the $\mathcal{F}$-$\mathbb{Z}$-algebra $\mathbb{Z}[\mathrm{GL}^\bullet_d]^{\mathrm{GL}_d}$ is generated by $\{s_1, \dots, s_d\}$ and $\det^{-1} = s_d^{-1}$ (see Theorem 4.11), the image of $\mathbb{Z}[\mathrm{GL}^\bullet_d]^{\mathrm{GL}_d}$ is contained $\mathcal{C}(\Gamma^\bullet, A)$, as desired. ☐

Corollary 6.6. Let Γ be a profinite group, let κ be a finite or local field and let Λ be a coefficient ring for κ. Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_{\mathrm{GL}_d}(\kappa)$ and let $D_{\overline{\Theta}}$ be the determinant law attached to $\overline{\Theta}$ by Theorem 6.4. Then the natural transformation $\mathrm{Def}_{\Lambda,\overline{\Theta}} \to \mathrm{Def}_{\Lambda,D_{\overline{\Theta}}}$ defined as in Proposition 6.5 is a natural bijection. In particular there is a canonical isomorphism $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \cong R^{\mathrm{ps}}_{\Lambda,D_{\overline{\Theta}}}$ of universal pseudodeformation rings.

Proof. This follows from Proposition 6.5 and Theorem 5.3. ☐

7. Comparing deformations and pseudodeformations

The main purpose of this section is to compare unframed deformation functors to pseudodeformation functors when the residue field of our deformation problem is a finite or a local field. We first prove a version of [BHKT16, Theorem 4.10] extended to local residue fields.

Proposition 7.1. Let Γ be a profinite group. Let κ be a finite or local field, let Λ be a coefficient ring for κ and let G be a connected reductive Λ-group scheme. Let $\overline{\rho} : \Gamma \to G(\kappa)$ be a continuous representation and let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma_F}_G(\kappa)$ be the associated pseudocharacter. Assume, that the centralizer of $\overline{\rho}$ is trivial in G^{ad} and that $\overline{\rho}$ is G-completely reducible. Then the natural map of deformation functors $\mathrm{Def}_{\Lambda,\overline{\rho}} \to \mathrm{Def}_{\Lambda,\overline{\Theta}}$ is an isomorphism.

Proof. Let $A \in \mathfrak{A}_\Lambda$ and $\Theta \in \mathrm{Def}_{\Lambda,\overline{\Theta}}(A)$. For any $n \geq 1$, we define affine Λ-schemes of finite type $X_n := G^n$ and $Y_n := G^n /\!\!/ G$ and let $\pi : X_n \to Y_n$ be the projection.

Now fix $n \geq 1$ and $\gamma_1, \dots, \gamma_n \in \Gamma$, such that the scheme-theoretic centralizer $Z_{G_\kappa}(x)$ of $x := (\overline{g}_1, \dots, \overline{g}_n)$ in G_κ coincides with the scheme-theoretic centralizer $Z_{G_\kappa}(\overline{\rho})$ of $\overline{\rho}$ in G_κ. This is possible, as $\kappa[G]$ is a noetherian ring. Thus the image of $Z_{G_\kappa}(x)$ in G_κ^{ad} is trivial by assumption. We may assume by [Mar03, Lemma 9.2], that the subgroup generated by $\overline{\rho}(\gamma_1), \dots, \overline{\rho}(\gamma_n)$ has the same Zariski closure as $\overline{\rho}(\Gamma)$, we denote this topological subgroup of $G(\kappa)$ by H. Since $\overline{\rho}$ is G_κ-completely reducible, by [BMR05, Proposition 2.16] the orbit of x in $X_{n,\kappa}$ is closed.

In [BHKT16, Theorem 4.10], the completion of X_n at $x \in X_n(\kappa)$ is defined as the functor $X_n^{\wedge,x} : \mathfrak{A}_\Lambda \to \mathrm{Set}$ defined by $X_n^{\wedge,x}(A) := X_n(A) \times_{X_n(\kappa)} \{x\}$. Similarly, for fixed $h \in H$, we define the completion of X_{n+1} at $y := (x,h) \in X_{n+1}(\kappa)$ and the respective completions of Y_n and Y_{n+1} at $\pi(x)$ and $\pi(y)$. Let $G^{\mathrm{ad},\wedge}$ be the completion of G^{ad} at the neutral element. It is a group functor on $\mathfrak{A}_\Lambda$, representable by a formal Λ-scheme.

In analogy to the completion at a point, we define the completion of X_{n+1} at H as the functor $X_{n+1}^{\wedge,H} : \mathfrak{A}_\Lambda \to \mathrm{Set}$ by $X_{n+1}^{\wedge,H}(A) := X_{n+1}(A) \times_{X_{n+1}(\kappa)} H$, where the map $H \to X_{n+1}(\kappa)$ is given by $h \mapsto (g_1, \dots, g_n, h)$. Similarly we define $Y_{n+1}^{\wedge,H}(A) := Y_{n+1}(A) \times_{Y_{n+1}(\kappa)} H$. We will need these completions to prove continuity of the representation we construct. One can think of completions at H just as putting the completions at single points of H into a continuous family.

Θ_{n+1} determines a natural map $\Lambda[G^{n+1}]^G \to \mathcal{C}(\Gamma, A),\ f \mapsto (\gamma \mapsto \Theta_{n+1}(f)(\gamma_1, \dots, \gamma_n, \gamma))$, which is an element $\alpha \in Y_{n+1}(\mathcal{C}(\Gamma, A)) = \mathcal{C}(\Gamma, Y_{n+1}(A))$. Here $Y_{n+1}(A)$ is endowed with the discrete topology if κ is finite and with the subspace topology of some closed immersion into an affine space over A equipped with the product topology as a κ-vector space.

By the universal property of pullbacks and compatibility of the topologies we have defined on point sets in Section 1.2, we obtain a unique continuous map $\beta : \Gamma \to Y_{n+1}^{\wedge,H}(A)$ as indicated in the diagram:

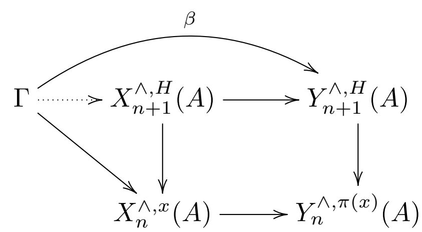
$$\begin{array}{ccccc}
\Gamma & \overset{\beta}{\dashrightarrow} & Y_{n+1}^{\wedge,H}(A) & \longrightarrow & Y_{n+1}(A) \\
& \searrow^{\overline{\rho}} & \downarrow & & \downarrow \\
& & H & \longrightarrow & Y_{n+1}(\kappa)
\end{array}
\qquad (\alpha : \Gamma \to Y_{n+1}(A))$$

The proof of [BHKT16, Proposition 3.13] goes through verbatim in our setting. Hence $G^{\mathrm{ad},\wedge}$ acts freely on $X_n^{\wedge,x}$ and the projection $X_n^{\wedge,x} \to Y_n^{\wedge,\pi(x)}$ factors through an isomorphism $X_n^{\wedge,x}/G^{\mathrm{ad},\wedge} \to Y_n^{\wedge,\pi(x)}$. In particular $X_n^{\wedge,x}(A) \to Y_n^{\wedge,\pi(x)}(A)$ is surjective and we can choose a preimage $(g_1, \dots, g_n) \in X_n^{\wedge,x}(A)$ of the point in $Y_n^{\wedge,\pi(x)}(A)$ determined by $\Lambda[G^n]^G \to A,\ f \mapsto \Theta_n(f)(\gamma_1, \dots, \gamma_n)$.

For fixed $h \in H$ and $y := (x,h)$, we have two cartesian squares:

$$\begin{array}{ccccc}
X_{n+1}^{\wedge,y}(A) & \longrightarrow & Y_{n+1}^{\wedge,\pi(y)}(A) & \longrightarrow & \{h\} \\
\downarrow & & \downarrow & & \downarrow \\
X_{n+1}^{\wedge,H}(A) & \longrightarrow & Y_{n+1}^{\wedge,H}(A) & \longrightarrow & H
\end{array}$$

As in the proof of [BHKT16, Theorem 4.10], the top left arrow is a $G^{\mathrm{ad},\wedge}(A)$-torsor of sets, so $X_{n+1}^{\wedge,y} \to Y_{n+1}^{\wedge,\pi(y)}$ is a $G^{\mathrm{ad},\wedge}$-pseudotorsor. It follows, that $X_{n+1}^{\wedge,H} \to Y_{n+1}^{\wedge,H}$ is a $G^{\mathrm{ad},\wedge}$-pseudotorsor. The square in the following diagram is cartesian, since the horizontal arrows are $G^{\mathrm{ad},\wedge}$-pseudotorsors and the vertical maps are equivariant:



The map $\Gamma \to X_n^{\wedge,x}(A)$ maps constantly to $(g_1, \dots, g_n)$. By the discussion of the topologies on point sets in Section 1.2, the diagram is also cartesian in the category of topological spaces. Again by the universal property, we obtain a continuous map $\Gamma \to X_{n+1}^{\wedge,G(\kappa)}(A)$.

The composition $\Gamma \to X_{n+1}^{\wedge,G(\kappa)}(A) \to X_{n+1}(A) \overset{\mathrm{pr}_{n+1}}{\to} G(A)$ defines the desired ρ with $\Theta_\rho = \Theta$ as in [BHKT16, Theorem 4.10]. The second map is continuous by definition of the completion of $X_{n+1}^{\wedge,G(\kappa)}$ as a pullback. The projection pr_{n+1} is continuous by definition of the topologies on point sets Proposition 1.6. So the composition ρ is continuous and this finishes the proof. □

We can now prove a continuous version of Theorem 2.3 for certain residual representations, which will be enough for the proof of Proposition 8.10.

Proposition 7.2. Let Γ be a profinite group, let κ be a finite or a local field, let Λ be a coefficient ring for κ and let G be a Chevalley group over Λ. Suppose $\Theta \in \mathrm{cPC}_G^\Gamma(\mathcal{O}_\kappa)$ is a continuous pseudocharacter, where $\mathcal{O}_\kappa = \kappa$ if κ is finite. If κ is a local field of positive characteristic, assume that Γ is topologically finitely generated and that the reduction $\overline{\Theta}$ of Θ to the residue field k of κ comes from a G-completely reducible representation $\overline{\rho} : \Gamma \to G(k')$ for some finite extension k'/k, which has scheme-theoretically trivial centralizer in $G_{k'}^{\mathrm{ad}}$. Then there exists a continuous (G-completely reducible) representation $\rho : \Gamma \to G(\overline{\kappa})$ with $\Theta_\rho = \Theta$, which is defined over the ring of integers $\mathcal{O}_{\kappa'}$ of a finite extension κ'/κ.

Proof. Suppose κ is finite. Then Theorem 2.3 provides us with a G-completely reducible representation $\rho : \Gamma \to G(\overline{\kappa})$, such that $\Theta_\rho = \Theta$. By [BHKT16, Proposition 4.7 (iii)], ρ is continuous. Since Γ is profinite, $\rho(\Gamma)$ is finite. In particular there exists a finite extension κ', such that $\rho(\Gamma) \subseteq G(\kappa')$.

If κ is a local field of characteristic 0, we can argue the same way using [BHKT16, Proposition 4.7 (ii)] and [BHKT16, Proposition 4.8 (ii)].

Assume κ is a local field of positive characteristic. Let k be the residue field of $\mathcal{O}_\kappa$. By the first step the reduction $\overline{\Theta}$ of Θ to k comes from a continuous G-completely reducible representation $\overline{\rho} : \Gamma \to G(k')$ over a finite extension k'/k, which by our assumption has scheme-theoretically trivial centralizer in $G_{k'}^{\mathrm{ad}}$. Choose a finite extension κ'/κ, such that the residue field of $\mathcal{O}_{\kappa'}$ is k'. So $\Theta \otimes_{\mathcal{O}_\kappa} \mathcal{O}_{\kappa'}$ is a deformation of $\overline{\Theta} \otimes_k k'$. By Proposition 7.1 $\Theta \otimes_{\mathcal{O}_\kappa} \mathcal{O}_{\kappa'}$ thus comes from a continuous deformation $\rho : \Gamma \to G(\mathcal{O}_{\kappa'})$ of $\overline{\rho}$. □

Definition 7.3. We say, that a prime p is *very good* for a simple algebraic group G over an algebraically closed field, if the following conditions hold.

(1) $p \nmid n+1$, if G is of type A_n.

(2) $p \neq 2$, if G is of type B, C, D, E, F, G.
(3) $p \neq 3$, if G is of type E, F, G.
(4) $p \neq 5$, if G is of type E_8.

We say, that p is *very good* for a reductive algebraic group G, if it is very good for every simple factor of G^0.

Lemma 7.4. Let Γ be a group. Let $G \subseteq \mathrm{GL}_n$ be a reductive group over an algebraically closed field k of characteristic $p \geq 0$ and let $\rho : \Gamma \to G(k)$ be a G-completely reducible representation, which is in addition irreducible after embedding into $\mathrm{GL}_n(k)$.

Assume, that one of the following holds:

(1) p is very good for G^{ad} and G^{ad} is connected.
(2) (GL_n, G) is a reductive pair, i.e. $\mathfrak{g}$ is a G-module direct summand of $\mathfrak{gl}_n$.

Then the scheme-theoretic centralizer $Z_{G^{\mathrm{ad}}}(\rho)$ of ρ in G^{ad} is trivial.

Proof. Beware, that $Z_{G^{\mathrm{ad}}}(\rho)$ is defined as follows. For $A \in \mathrm{CAlg}_k$, the group $Z_{G^{\mathrm{ad}}}(\rho)(A)$ is defined as the kernel of the map

$$G^{\mathrm{ad}}(A) \to \mathrm{Hom}(\Gamma, G(A)), \quad g \mapsto g\rho g^{-1}$$

By Schur's lemma $Z_{\mathrm{GL}_n}(\rho)(k) = Z(\mathrm{GL}_n)(k)$. Let $\pi : G \twoheadrightarrow G^{\mathrm{ad}}$ be the canonical projection. By definition, $Z_G(\rho) = \pi^{-1}(Z_{G^{\mathrm{ad}}}(\rho))$ and $Z_G(\rho) = Z_{\mathrm{GL}_n}(\rho) \cap G$. We get $Z(G)(k) \subseteq \pi^{-1}(Z_{G^{\mathrm{ad}}}(\rho)(k)) = Z_G(\rho)(k) = Z_{\mathrm{GL}_n}(\rho)(k) \cap G(k) \subseteq Z(G)(k)$. We conclude, that $Z_{G^{\mathrm{ad}}}(\rho)(k)$ is trivial.

Assuming (1), we see by [BMRT07, Theorem 1.2] since p is very good for G^{ad} and G^{ad} is connected, that $Z_{G^{\mathrm{ad}}}(\rho)$ is smooth and thus trivial as an algebraic group.

Assuming (2), we obtain from [BMRT07, Corollary 2.13], that $Z_G(\rho)$ is smooth. Since GL_n is separable, $Z_{\mathrm{GL}_n}(\rho)$ is also smooth and we have $Z_{\mathrm{GL}_n}(\rho) = Z(\mathrm{GL}_n)$. We can repeat the above calculation without taking points:

$$Z(G) \subseteq \pi^{-1}(Z_{G^{\mathrm{ad}}}) = Z_G(\rho) = Z_{\mathrm{GL}_n}(\rho) \cap G = Z(\mathrm{GL}_n) \cap G \subseteq Z(G)$$

Hence $Z_{G^{\mathrm{ad}}} = 1$. □

Proposition 7.5. Let $\overline{\rho} : \Gamma_F \to G(\kappa)$ be a continuous representation over a finite or local field κ and let Λ be a coefficient ring for κ. Assume, that the unframed deformation functor is representable by $R_{\overline{\rho}}$. We have a presentation $R_{\overline{\rho}} \cong \Lambda[[x_1, \dots, x_r]]/(f_1, \dots, f_s)$, where $r = h^1(\Gamma_F, \mathrm{ad}_{\overline{\rho}})$ and $s = h^2(\Gamma_F, \mathrm{ad}_{\overline{\rho}})$.

Proof. This follows from a standard calculation with cocycles. See e.g. [Til96]. □

Proposition 7.6. Let F be a p-adic local field with absolute Galois group Γ_F. Let κ be a finite or local field of very good characteristic $p \geq 0$ for $G^{\mathrm{ad}}_{\overline{\kappa}}$, let Λ be a coefficient ring for κ and let $G \subseteq \mathrm{GL}_n$ be a Chevalley group over Λ. Let $\overline{\rho} : \Gamma_F \to G(\kappa)$ be an absolutely G-completely reducible continuous representation with associated G-pseudocharacter $\overline{\Theta} \in \mathrm{cPC}^{\Gamma_F}_G(\kappa)$, such that $\overline{\rho}$ is absolutely irreducible after embedding into $\mathrm{GL}_n(\overline{\kappa})$ and such that $H^2(\Gamma_F, \mathfrak{g}_\kappa) = 0$.

Assume, that one of the following holds:

(1) p is very good for $G^{\mathrm{ad}}_{\overline{\kappa}}$ and $G^{\mathrm{ad}}_{\overline{\kappa}}$ is connected.
(2) $(\mathrm{GL}_{n,\overline{\kappa}}, G_{\overline{\kappa}})$ is a reductive pair, i.e. $\mathfrak{g}_{\overline{\kappa}}$ is a $G_{\overline{\kappa}}$-module direct summand of $\mathfrak{gl}_{n,\overline{\kappa}}$.

Then $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$ is formally smooth over Λ of dimension $\dim \mathfrak{g}_\kappa \cdot [F : \mathbb{Q}_p] + h^0(\Gamma_F, \mathfrak{g}_\kappa) + \dim \Lambda$. In particular $R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \cong \Lambda[[x_1, \dots, x_r]]$.

Proof. By Lemma 7.4 the scheme-theoretic centralizer of $\overline{\rho}$ in $G^{\mathrm{ad}}_{\overline{\kappa}}$ is trivial. We can apply Proposition 7.1 to obtain a canonical isomorphism $R_{\overline{\rho}} \cong R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}}$. By Proposition 7.5, the deformation ring $R_{\overline{\rho}}$ is isomorphic to $\Lambda[[x_1, \dots, x_r]]$, where $r = h^1(\Gamma_F, \mathfrak{g}_\kappa)$. The Euler characteristic formula [BJ19, Theorem 3.4.1] implies, that $\dim R_{\overline{\rho}} = \dim \mathfrak{g}_\kappa \cdot [F : \mathbb{Q}_p] + h^0(\Gamma_F, \mathfrak{g}_\kappa) + \dim \Lambda$. □

8. Dimension of $R^{\mathrm{ps}}_{\overline{\Theta}}$

Let $\mathcal{O}_L$ be the ring of integers of a p-adic field L with uniformizer ϖ and residue field κ, let G be a Chevalley group over $\mathcal{O}_L$ and let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_G(\kappa)$ be a continuous G-pseudocharacter. Let $\overline{X}_{\overline{\Theta}} := \mathrm{Spec}(R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}/\varpi)$, where $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is the universal pseudodeformation ring of $\overline{\Theta}$ with coefficients $\mathcal{O}$ from Theorem 5.3. We define

$$\mathrm{Sp}_{2n}(A) := \{M \in \mathrm{GL}_{2n}(A) \mid M^{-1} = JM^\top J^{-1}\},$$

where $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$ for every commutative ring A. In this section, we use the methods developed in [BJ19] to estimate the dimension of $\overline{X}_{\overline{\Theta}}$ for $G = \mathrm{Sp}_{2n}$ and $\Gamma = \Gamma_F$ the absolute Galois group of a local field $F/\mathbb{Q}_p$. We assume throughout, that $p > 2$. Note, that by Theorem 5.9 $R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is noetherian, since Γ_F is topologically finitely generated. In lack of reference for this fact, we refer to Chandan Singh Dalawat's answer to Mathoverflow Question # 63094. Let $\iota : \mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$ be the standard representation. By Proposition 6.5, the GL_{2n}-pseudocharacter $\iota(\overline{\Theta})$ corresponds to a unique determinant law $D_{\iota(\overline{\Theta})}$ of dimension $2n$. The pseudodeformation ring $R^{\mathrm{univ}}_{\mathcal{O}_L,D_{\iota(\overline{\Theta})}}$ of $D_{\iota(\overline{\Theta})}$ defined in [BJ19, Proposition 4.7.4] is by Corollary 6.6 canonically isomorphic to $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})}$. We shall use this identification without further mention whenever we cite results from [BJ19].

8.1. **Symplectic representations.** A *symplectic representation* V of a group Γ over a field k is a representation, that carries a Γ-invariant antisymmetric non-degenerate bilinear form $\beta : V \times V \to k$. When $p > 2$, two semisimple symplectic representations over an algebraically closed field are conjugate over Sp_{2n} if and only if they are conjugate over GL_{2n}. This is a consequence of the fact, that when $p > 2$ the notions of Sp_{2n}-semisimplicity and GL_{2n}-semisimplicity coincide [Ric88, Corollary 16.10] and the uniqueness part of Theorem 2.3. So being symplectic can be seen as a property of GL_{2n}-conjugacy classes of semisimple representations. It is easy to check, that a representation of the form $W \oplus W^*$ for some arbitrary representation W is always symplectic. We call these *representations of pair type*. In general a semisimple symplectic representation is a direct sum of irreducible symplectic representations and representations of pair type.

Proposition 8.1. Every semisimple symplectic representation of a group Γ over an algebraically closed field k is a direct sum of representations of one of the following two types.

(1) An irreducible symplectic representation.
(2) A direct sum $V \oplus V^*$, where V is an arbitrary irreducible representation.

Proof. Let V be a symplectic representation.

We proceed by induction over $\dim V$. If $\dim V = 0$ there is nothing to show. We assume $\dim V > 0$. Let W be an irreducible subrepresentation of V and assume, that $\beta : W \times W \to k$ is non-degenerate. In particular W is an irreducible symplectic representation. Then $W^\perp$ is non-degenerate and Γ-invariant and we may assume $W^\perp$ has the desired form. This implies the claim.

We now assume, that V has no irreducible subrepresentation on which β is non-degenerate. Let W be any irreducible subrepresentation of V. Since β is non-degenerate, there is an irreducible subrepresentation $W' \neq W$, such that $\beta : W \times W' \to k$ is non-degenerate. β is non-degenerate on $W \oplus W'$, so $(W \oplus W')^{\perp}$ is non-degenerate and Γ-invariant. As in the previous case, this implies the claim. □

This motivates the following terminology. We say that a symplectic representation V is *symplectically decomposable*, if it can be written as the direct sum of two nonzero symplectic representations, and *symplectically indecomposable* otherwise. There are exactly two types of symplectically indecomposable representations: Those which are irreducible under the standard embedding into GL_{2n} and those which are a direct sum $W \oplus W^*$ for some irreducible representation W.

8.2. **Subdivision of $\overline{X}_{\overline{\Theta}}$.** For a point $x \in \overline{X}_{\overline{\Theta}}$, there is a natural G-pseudocharacter $\Theta_x \in \mathrm{PC}_G^{\Gamma}(\overline{\kappa(x)})$ defined after choice of an algebraic closure $\overline{\kappa(x)}$ of the residue field $\kappa(x)$ of x. Let P be a property of G-completely reducible representations over an algebraically closed field, which is stable under G-conjugation and passage to algebraically closed sub- and overfields. We say x has property P, if the G-completely reducible representation attached to Θ_x by Theorem 2.3 has property P. If Q is a property of representations into GL_{2n}, we say that a representation ρ into Sp_{2n} has property Q, if ρ followed by the standard representation $\iota : \mathrm{Sp}_{2n} \to \mathrm{GL}_{2n}$ has property Q.

In their analysis [BJ19] of the special fiber of the pseudodeformation space for GL_n, Böckle and Juschka have noticed that irreducible points need not be unobstructed. They have found a convenient characterization of obstructed irreducible points [BJ19, Lemma 5.1.1], which allows them to find good dimension bounds for the obstructed locus. We recall their definition [BJ19, Definition 5.1.2]. It turns out, that for $G = \mathrm{Sp}_{2n}$ the dimension of the locus of special points for GL_{2n} is still small enough to get the desired estimates.

Definition 8.2. Let k be an algebraically closed $\mathbb{Z}_p$-field and let $\rho : \Gamma_F \to \mathrm{GL}_{2n}(k)$ be an irreducible representation. We say, that ρ is *special*, if one of the following holds.

(1) $\zeta_p \notin F$ and $\rho \cong \rho(1)$.
(2) $\zeta_p \in F$ and there is some degree p Galois extension F'/F, such that $\rho|_{\Gamma_{F'}}$ is reducible.

Definition 8.3. We define the following subsets of $\overline{X}_{\overline{\Theta}}$.

(1) $\overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}}$ is the subset of non-special points.
(2) $\overline{X}_{\overline{\Theta}}^{\mathrm{spcl}}$ is the subset of special points.
(3) $\overline{X}_{\overline{\Theta}}^{\mathrm{pair}}$ is the subset of points of pair type.
(4) $\overline{X}_{\overline{\Theta}}^{\mathrm{dec}}$ is the subset of symplectically decomposable points.
(5) For any of the above subsets $\dot{\overline{X}}_{\overline{\Theta}}^{?} := \overline{X}_{\overline{\Theta}}^{?} \setminus \{\mathfrak{m}_{R^{\mathrm{ps}}_{\mathcal{O},\overline{\Theta}}}\}$.

Proposition 8.4. $\overline{X}_{\overline{\Theta}} = \overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}} \,\dot\cup\, \overline{X}_{\overline{\Theta}}^{\mathrm{spcl}} \,\dot\cup\, (\overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{pair}})$.

Proof. This follows directly from Proposition 8.1. □

Lemma 8.5. Suppose $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_2 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2n}}(\kappa)$ with $\overline{\Theta}_1 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2a}}(\kappa)$, $\overline{\Theta}_2 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2b}}(\kappa)$ and $a+b=n$, where the direct sum is a direct sum of symplectic pseudocharacters as explained in Section 2.3. Then the map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\overline{\Theta}_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\overline{\Theta}_2}$ induced by $(\Theta_1, \Theta_2) \mapsto \Theta_1 \oplus \Theta_2$ is finite.

Proof. Let $\iota : \mathrm{Sp}_{2d} \to \mathrm{GL}_{2d}$ be the canonical embedding and let $\iota(\overline{\Theta})$ be the associated GL_{2d}-pseudocharacter, similarly for $\iota_i(\overline{\Theta}_i)$. By Lemma 5.11 $R^{\mathrm{ps}}_{\overline{\Theta}}$ is a quotient of $R^{\mathrm{ps}}_{\iota(\overline{\Theta})}$ and similarly for $R^{\mathrm{ps}}_{\overline{\Theta}_i}$. We know from [BIP21, Lemma 3.23], that the map $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)}$ is finite. It follows, that the induced map $R^{\mathrm{ps}}_{\overline{\Theta}} \to (R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)}) \widehat{\otimes}_{R^{\mathrm{ps}}_{\iota(\overline{\Theta})}} R^{\mathrm{ps}}_{\overline{\Theta}}$ is finite. Since there is a natural surjection $R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)} \twoheadrightarrow R^{\mathrm{ps}}_{\overline{\Theta}_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\overline{\Theta}_2}$, the natural map $(R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_2(\overline{\Theta}_2)}) \widehat{\otimes}_{R^{\mathrm{ps}}_{\iota(\overline{\Theta})}} R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\overline{\Theta}_1} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\overline{\Theta}_2}$ is surjective. □

Lemma 8.6. Let $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_1^* \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2n}}(\kappa)$ be a symplectic representation as explained at the end of Section 2.3 with $\overline{\Theta}_1 \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{GL}_n}(\kappa)$. Then the map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\overline{\Theta}_1}$ induced by $\Theta_1 \mapsto \Theta_1 \oplus \Theta_1^*$ is finite.

Proof. As in the proof of Lemma 8.5, the map $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)}$ is finite. By affineness, the map $R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{\mathcal{O}} R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)}$ induced by $\Theta_1 \mapsto (\Theta_1, \Theta_1^*)$ is surjective. So the composition $R^{\mathrm{ps}}_{\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)}$ is finite and induced by $\Theta_1 \mapsto \Theta_1 \oplus \Theta_1^*$. Tensoring with $R^{\mathrm{ps}}_{\overline{\Theta}}$, we obtain a finite map $R^{\mathrm{ps}}_{\overline{\Theta}} \to R^{\mathrm{ps}}_{\iota_1(\overline{\Theta}_1)} \widehat{\otimes}_{R^{\mathrm{ps}}_{\iota(\overline{\Theta})}} R^{\mathrm{ps}}_{\overline{\Theta}} \cong R^{\mathrm{ps}}_{\overline{\Theta}_1}$. The last isomorphism can be seen to hold by considering the corresponding deformation functors. □

Proposition 8.7.

(1) The natural map $\overline{X}_{\overline{\Theta}} \to \overline{X}_{\iota(\overline{\Theta})}$ is a closed immersion.
(2) $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ is closed in $\overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$.
(3) $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$ is closed in $\overline{X}_{\overline{\Theta}}$.
(4) $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$ is closed in $\overline{X}_{\overline{\Theta}}$.
(5) $\overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}}$ is open in $\overline{X}_{\overline{\Theta}}$.

Proof.

(1) By Theorem 5.12, the map $R^{\mathrm{ps}}_{\mathcal{O}_L,\iota(\overline{\Theta})} \to R^{\mathrm{ps}}_{\mathcal{O}_L,\overline{\Theta}}$ is surjective.
(2) $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ is the preimage of $\overline{X}^{\mathrm{spcl}}_{\iota(\overline{\Theta})}$ under the closed immersion $\overline{X}_{\overline{\Theta}} \to \overline{X}_{\iota(\overline{\Theta})}$. Since $\overline{X}^{\mathrm{nspcl}}_{\iota(\overline{\Theta})}$ is open in $\overline{X}_{\iota(\overline{\Theta})}$ by [BJ19, Theorem 4.5.1 (ii)], $\overline{X}^{\mathrm{spcl}}_{\iota(\overline{\Theta})}$ is closed in $\overline{X}^{\mathrm{irr}}_{\iota(\overline{\Theta})}$ and the claim follows.
(3) $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}}$ is the union of the images of finitely many maps as in Lemma 8.6.
(4) $\overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$ is the union of the images of finitely many maps as in Lemma 8.5.
(5) $\overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$ is open in $\overline{X}_{\overline{\Theta}}$, as the complement of $\overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \cup \overline{X}^{\mathrm{dec}}_{\overline{\Theta}}$ (see Proposition 8.4). The subset $\overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}} \subseteq \overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$ is the complement of $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$, which is closed in $\overline{X}^{\mathrm{irr}}_{\overline{\Theta}}$. Hence $\overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}}$ is open in an open subset of $\overline{X}_{\overline{\Theta}}$.

□

Lemma 8.8. Let $\bar{f} : \kappa \to \kappa'$ be a homomorphism between either two finite or two local fields. Let $f : \Lambda \to \Lambda'$ be a local homomorphism of complete noetherian local rings with residue fields κ and κ' respectively and assume, that f reduces to $\bar{f}$ on residue fields. Let Γ be a profinite group and let G be an affine Λ-group scheme. Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma}_{G}(\kappa)$ and define $\overline{\Theta}' := \overline{\Theta} \otimes_{\kappa} \kappa'$. Then the natural map

$$R^{\mathrm{ps}}_{\Lambda',\overline{\Theta}'} \to R^{\mathrm{ps}}_{\Lambda,\overline{\Theta}} \widehat{\otimes}_{\Lambda} \Lambda'$$

induced by

$$\mathrm{Def}_{\Lambda,\overline{\Theta}}(A) \to \mathrm{Def}_{\Lambda',\overline{\Theta}'}(A \otimes_\Lambda \Lambda'), \quad \Theta \mapsto \Theta \otimes_\Lambda \Lambda'; \qquad A \in \mathfrak{A}_\Lambda$$

is an isomorphism.

Proof. The proof of [BJ19, Proposition 4.7.6] carries over in our setting. □

8.3. **Dimension bounds for $G = \mathrm{Sp}_{2n}$.** The following proposition is the analog of [BJ19, Lemma 5.1.6] for $G = \mathrm{Sp}_{2n}$.

Lemma 8.9. Let k be a field with $2 \in k^\times$. Then the symplectic Lie algebra $\mathfrak{sp}_{2n,k}$ is a direct summand of $\mathfrak{gl}_{2n,k}$ and of $\mathfrak{sl}_{2n,k}$ and the corresponding projection maps $\mathfrak{gl}_{2n,k} \twoheadrightarrow \mathfrak{sp}_{2n,k}$ and $\mathfrak{sl}_{2n,k} \twoheadrightarrow \mathfrak{sp}_{2n,k}$ are equivariant for the adjoint action of the symplectic group Sp_{2n}.

Proof. Recall, that $\mathfrak{sp}_{2n,k} = \{M \in \mathfrak{gl}_{2n,k} \mid JM^\top + MJ = 0\}$, where $J = \left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$. Right multiplication with J is an isomorphism of k-vector spaces $- \cdot J : \mathfrak{gl}_{2n,k} \to M_{2n}(k)$ and identifies $\mathfrak{sp}_{2n,k}$ with the subspace of symmetric $2n \times 2n$ matrices. The symmetrization map $a : M_{2n}(k) \to M_{2n}(k),\ M \mapsto \frac{1}{2}(M + M^\top)$ shows, that symmetric matrices are a direct summand of $M_{2n}(k)$. The map $\mathfrak{gl}_{2n}(k) \to \mathfrak{gl}_{2n}(k),\ M \mapsto a(MJ)J^{-1}$ is equivariant for the adjoint action of Sp_{2n} on $\mathfrak{gl}_{2n}(k)$: Suppose $M \in M_{2n}(k)$ and $A \in \mathrm{Sp}_{2n}(k)$: Then

$$a(AMA^{-1}J)J^{-1} = \frac{1}{2}(AMA^{-1} + J^{-1}(A^{-1})^\top M^\top A^\top J^{-1})$$

and

$$Aa(MJ)J^{-1}A^{-1} = \frac{1}{2}(AMA^{-1} + AJ^{-1}M^\top J^{-1}A^{-1}) = \frac{1}{2}(AMA^{-1} + J^{-1}(A^{-1})^\top M^\top A^\top J^{-1})$$

using $A \in \mathrm{Sp}_{2n}(k)$, so that $A^{-1} = JA^\top J^{-1}$ and $J^\top = J^{-1}$. We also obtain, that the projection map $\mathfrak{gl}_{2n,k} \twoheadrightarrow \mathfrak{sp}_{2n,k}$ is split by the inclusion and equivariant for the adjoint action of Sp_{2n}. Since $\mathfrak{sp}_{2n,k} \subseteq \mathfrak{sl}_{2n,k}$, the restriction $\mathfrak{sl}_{2n,k} \to \mathfrak{sp}_{2n,k}$ is still split by the inclusion and Sp_{2n}-equivariant. □

Proposition 8.10. Let $\overline{\Theta} \in \mathrm{cPC}^{\Gamma_F}_{\mathrm{Sp}_{2n}}(\kappa)$ with κ a finite field of characteristic $p > 2$ and let Λ be a coefficient ring for κ. Let $x \in U := \overline{X}^{\mathrm{irr}}_{\Lambda,\overline{\Theta}}$ be a closed point. By [BIP21, Lemma 3.16] the residue field $\kappa(x)$ of x is a local field. Let $R^{\mathrm{ps}}_{\Theta_x}$ be the universal pseudodeformation ring of the Sp_{2n}-pseudocharacter Θ_x attached to x. By Proposition 7.2, there is a finite extension κ' of $\kappa(x)$, such that $\Theta'_x := \Theta_x \otimes_{\kappa(x)} \kappa'$ is induced by a continuous absolutely irreducible representation $\overline{\rho} : \Gamma_F \to G(\kappa')$.

(1) (a) Suppose, that x is non-special. Then $R^{\mathrm{ps}}_{\Theta'_x}$ is regular of dimension $n(2n+1) \cdot [F : \mathbb{Q}_p]$.
    (b) If in addition $U^{\mathrm{nspcl}} \neq \emptyset$, then U^{nspcl} is regular and equidimensional of dimension $n(2n+1) \cdot [F : \mathbb{Q}_p] - 1$.
(2) Suppose, that $\zeta_p \notin F$ and that x is special. Then $\dim R^{\mathrm{ps}}_{\Theta'_x} \in \{n(2n+1) \cdot [F : \mathbb{Q}_p], n(2n+1) \cdot [F : \mathbb{Q}_p] + 1\}$.
(3) If $\zeta_p \notin F$, then $\dim U \leq n(2n+1) \cdot [F : \mathbb{Q}_p]$.

Proof. Ad (1) (a). If $\zeta_p \notin F$, then by [BJ19, Lemma 5.1.1 Case I], we have $H^2(\Gamma_F, \mathfrak{gl}_{2n,\kappa'}) = 0$. Since 2 is invertible in κ', by Lemma 8.9 $\mathfrak{sp}_{2n,\kappa'}$ is a direct summand of $\mathfrak{gl}_{2n,\kappa'}$ and so $H^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$. If $\zeta_p \in F$, then we have $H^2(\Gamma_F, \mathfrak{sl}_{2n}) = 0$ by [BJ19, Lemma 5.1.1 case II]. By Lemma 8.9, $\mathfrak{sp}_{2n}$ is also a direct summand of $\mathfrak{sl}_{2n}$. It follows, that $H^2(\Gamma_F, \mathfrak{sp}_{2n}) = 0$. Let $R^{\mathrm{ps}}_{\Theta'_x}$ be the universal pseudodeformation ring of Θ'_x over a coefficient ring $\Lambda' \supseteq \Lambda$ with residue field κ'. By Proposition 7.6 $R^{\mathrm{ps}}_{\Theta'_x}$ is regular of dimension

$\dim \mathfrak{sp}_{2n,\kappa'} \cdot [F:\mathbb{Q}_p] + h^0(\Gamma_F, \mathfrak{sp}_{2n,\kappa'})$. By Schur's lemma $h^0(\Gamma_F, \mathfrak{gl}_{2n,\kappa'}) = 1$. Clearly $H^0(\Gamma_F, \mathfrak{gl}_{2n,\kappa'})$ is spanned by the diagonal matrices in $\mathfrak{gl}_{2n,\kappa'}$. These are not contained in $\mathfrak{sp}_{2n,\kappa'}$, hence $h^0(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$.

Ad (1) (b). Assume, that x is non-special. By Proposition 5.4, the universal pseudodeformation ring $R^{\mathrm{ps}}_{\Theta_x}$ can be identified with the completion of $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_\Lambda \kappa(x)$ at the kernel of the natural map $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_\Lambda \kappa(x) \to \kappa(x)$ attached to x. Since x is a 1-dimensional point of $R^{\mathrm{ps}}_{\overline{\Theta}}$ with residue characteristic p, it follows from [BJ19, Lemma 3.3.3], that x is a regular point of dimension $n(2n+1)\cdot[F:\mathbb{Q}_p]-1$ of U^{nspcl}. Let $U^{\mathrm{sing}} \subseteq U^{\mathrm{nspcl}}$ be the closed subscheme of singular points. By [Sta22, 02J4] and [Sta22, 01TB], the closed points are dense in U^{sing}. But since all closed points of U^{nspcl} are regular, U^{sing} must be empty. Since closed points are dense in U^{nspcl}, it follows that U^{nspcl} is equidimensional of dimension $n(2n+1)\cdot[F:\mathbb{Q}_p]-1$.

Ad (2). As in (1)(a) $h^0(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$. Since x is special, we have $\overline{\rho} \cong \overline{\rho}(1)$ by [BJ19, Lemma 5.1.1 Case (I)]. We have $H^2(\Gamma_F, \mathfrak{gl}_{2n,\kappa'}) \cong \mathrm{Hom}_{\Gamma_F}(\overline{\rho}, \overline{\rho}(1)) \cong \kappa'$ since $\overline{\rho}$ is irreducible, hence $h^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) \leq 1$. The case when $h^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 0$ is already covered in Proposition 7.6, so we assume $h^2(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = 1$. By the Euler characteristic formula [BJ19, Theorem 3.4.1]

$$h^1(\Gamma_F, \mathfrak{sp}_{2n,\kappa'}) = n(2n+1)[F:\mathbb{Q}_p]+1$$

and by Proposition 7.5, $R^{\mathrm{ps}}_{\Theta'_x}$ is a quotient of $\kappa'[[x_1,\dots,x_{n(2n+1)[F:\mathbb{Q}_p]+1}]]$ by an ideal generated by at most one element, so the assertion follows.

Ad (4). Let $x \in U$ be a closed point. Cases (1) and (2) imply, that $\dim R^{\mathrm{ps}}_{\Theta_x} \leq n(2n+1)[F:\mathbb{Q}_p]+1$. As in (1)(b), identifying $R^{\mathrm{ps}}_{\Theta_x}$ with a completion of $R^{\mathrm{ps}}_{\overline{\Theta}} \otimes_\Lambda \kappa(x)$ and applying [BJ19, Lemma 3.3.3], we see that U has dimension $\leq n(2n+1)[F:\mathbb{Q}_p]$. □

Proposition 8.11. Assume $G = \mathrm{Sp}_{2n}$. Then $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F:\mathbb{Q}_p]+1$. In particular if $n[F:\mathbb{Q}_p] \geq 3$ and if $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq \dim \overline{X}_{\overline{\Theta}} - 2$.

Proof. Since $\overline{X}^{\mathrm{spcl}}_{\overline{\Theta}}$ is a closed subspace of $\overline{X}^{\mathrm{spcl}}_{\iota(\overline{\Theta})}$ by Proposition 8.7 and the latter can be identified with the special locus of the pseudodeformation space of the determinant law $\overline{D}$ attached to $\iota(\overline{\Theta})$ by Theorem 6.4, we can take the estimate [BJ19, Theorem 5.3.1 (i)] to obtain $\dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \leq 2n^2[F:\mathbb{Q}_p]+1$. If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then $\dim \overline{X}_{\overline{\Theta}} \geq \dim \overline{X}^{\mathrm{nspcl}}_{\overline{\Theta}} = n(2n+1)[F:\mathbb{Q}_p]$ by Proposition 8.10 (1)(b). We get $\dim \overline{X}_{\overline{\Theta}} - \dim \overline{X}^{\mathrm{spcl}}_{\overline{\Theta}} \geq n[F:\mathbb{Q}_p]-1 \geq 2$. □

Theorem 8.12. Assume $G = \mathrm{Sp}_{2n}$.

(1) $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq n(2n+1)[F:\mathbb{Q}_p] - 4(n-1)[F:\mathbb{Q}_p]$.
In particular, if $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then $\dim \overline{X}^{\mathrm{dec}}_{\overline{\Theta}} \leq \dim \overline{X}_{\overline{\Theta}} - 4$.
(2) $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq n^2[F:\mathbb{Q}_p]+1$.
In particular, if $\overline{X}_{\overline{\Theta}}$ contains a non-special point and $n[F:\mathbb{Q}_p] \geq 2$, then $\dim \overline{X}^{\mathrm{pair}}_{\overline{\Theta}} \leq \dim \overline{X}_{\overline{\Theta}} - 3$.
(3) $\dim \overline{X}_{\overline{\Theta}} \leq n(2n+1)[F:\mathbb{Q}_p]$.
In particular, if $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then equality holds.

Proof. We make an induction over n, so we assume the entire theorem has been proved for all $n' < n$. Since our assertions are only about dimensions, by Lemma 8.8 we may assume that $\iota(\overline{\Theta})$ comes from a representation $\Gamma_F \to \mathrm{GL}_{2n}(\kappa)$ and that the irreducible constituents are absolutely irreducible.

(1) If $n = 1$, then the decomposable locus $\overline{X}_{\overline{\Theta}}^{\mathrm{dec}}$ is empty, so we may assume $n \geq 2$. There are up to isomorphism only finitely many ways to write $\overline{\Theta}$ as a direct sum of two symplectic pseudocharacters $\overline{\Theta}_1 \oplus \overline{\Theta}_2$. Here the notion of direct sum is that for symplectic pseudocharacters, introduced in Section 2.3. By Lemma 8.5, the map

$$\iota_{\overline{\Theta}_1,\overline{\Theta}_2}^{\mathrm{dec}} : \overline{X}_{\overline{\Theta}_1} \widehat{\times}_{\mathcal{O}} \overline{X}_{\overline{\Theta}_2} \to \overline{X}_{\overline{\Theta}}$$

is finite. We have an inclusion

$$\overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \subseteq \bigcup_{\overline{\Theta}_1 \oplus \overline{\Theta}_2 = \overline{\Theta}} \iota_{\overline{\Theta}_1,\overline{\Theta}_2}^{\mathrm{dec}}(\overline{X}_{\overline{\Theta}_1} \widehat{\times}_{\mathcal{O}} \overline{X}_{\overline{\Theta}_2})$$

where the right hand side is a closed subset of $\overline{X}_{\overline{\Theta}}$. Suppose $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_2$ is a decomposition into an Sp_{2a}-pseudocharacter $\overline{\Theta}_1$ and an Sp_{2b}-pseudocharacter $\overline{\Theta}_2$ for $a + b = n$ with $a, b \geq 1$. Then since $\iota_{\overline{\Theta}_1,\overline{\Theta}_2}^{\mathrm{dec}}$ is finite and by part (3) of the inductive hypothesis, we have

$$\begin{aligned} \dim \iota_{\overline{\Theta}_1,\overline{\Theta}_2}^{\mathrm{dec}}(\overline{X}_{\overline{\Theta}_1} \widehat{\times} \overline{X}_{\overline{\Theta}_2}) &\leq \dim(\overline{X}_{\overline{\Theta}_1} \widehat{\times} \overline{X}_{\overline{\Theta}_2}) \\ &\leq a(2a+1)[F : \mathbb{Q}_p] + b(2b+1)[F : \mathbb{Q}_p] \end{aligned}$$

Calculating

$$\begin{aligned} &n(2n+1)[F : \mathbb{Q}_p] - a(2a+1)[F : \mathbb{Q}_p] - b(2b+1)[F : \mathbb{Q}_p] \\ &= 4ab[F : \mathbb{Q}_p] \geq (\min_{\substack{a+b=n \\ a,b\geq 1}} 4ab) \cdot [F : \mathbb{Q}_p] = 4(n-1)[F : \mathbb{Q}_p] \end{aligned}$$

we obtain the desired bound

$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \leq n(2n+1)[F : \mathbb{Q}_p] - 4(n-1)[F : \mathbb{Q}_p]$$

If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, then by Proposition 8.10 (1)(b), we have a lower bound $\dim \overline{X}_{\overline{\Theta}} \geq n(2n+1)[F : \mathbb{Q}_p]$. Since $4(n-1)[F : \mathbb{Q}_p] \geq 4$, this implies the assertion.

(2) There are finitely many ways to write $\overline{\Theta} = \overline{\Theta}_1 \oplus \overline{\Theta}_1^*$ for some GL_n-pseudocharacter $\overline{\Theta}_1$ and we may assume, that there is at least one way. The sum yields an Sp_{2n}-pseudocharacter, as explained in Section 2.3. By Lemma 8.6, the map

$$\iota_{\overline{\Theta}_1}^{\mathrm{pair}} : \overline{X}_{\overline{\Theta}_1} \to \overline{X}_{\overline{\Theta}}$$

induced by $\Theta_1 \mapsto \Theta_1 \oplus \Theta_1^*$ is finite. We have an inclusion

$$\overline{X}_{\overline{\Theta}}^{\mathrm{pair}} \subseteq \bigcup_{\overline{\Theta}_1 \oplus \overline{\Theta}_1^* = \overline{\Theta}} \iota_{\overline{\Theta}_1}^{\mathrm{pair}}(\overline{X}_{\overline{\Theta}_1})$$

and the estimate

$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{pair}} \leq \dim \overline{X}_{\overline{\Theta}_1} = n^2[F : \mathbb{Q}_p] + 1$$

where the last equality follows from [BJ19, 5.4.1] after applying the bijection Corollary 6.6. If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, we obtain a lower bound as in step (1) and the estimate $n(n+1)[F : \mathbb{Q}_p] - 1 \geq 3$ implies the assertion.

(3) Let us recollect all upper bounds, we have established.

$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}} \overset{8.10\ (1)(\mathrm{b})}{\leq} n(2n+1)\cdot[F:\mathbb{Q}_p]$$
$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{spcl}} \overset{8.11}{\leq} 2n^2\cdot[F:\mathbb{Q}_p]+1$$
$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \overset{(1)}{\leq} n(2n+1)[F:\mathbb{Q}_p]-4(n-1)[F:\mathbb{Q}_p]$$
$$\dim \overline{X}_{\overline{\Theta}}^{\mathrm{pair}} \overset{(2)}{\leq} n^2[F:\mathbb{Q}_p]+1$$

Using the stratification $\overline{X}_{\overline{\Theta}} = \overline{X}_{\overline{\Theta}}^{\mathrm{nspcl}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{spcl}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{dec}} \cup \overline{X}_{\overline{\Theta}}^{\mathrm{pair}}$ from Proposition 8.4, we obtain the desired dimension bound for $\overline{X}_{\overline{\Theta}}$. If $\overline{X}_{\overline{\Theta}}$ contains a non-special point, we obtain equality from Proposition 8.10 (1)(b).

□

Corollary 8.13. Assume $G = \mathrm{Sp}_{2n}$ and that $\overline{\Theta}$ comes from a residual representation $\overline{\rho} : \Gamma_F \to \mathrm{Sp}_{2n}(\kappa)$, which is absolutely irreducible under the standard embedding into $\mathrm{GL}_{2n}(\kappa)$. Then $\dim \overline{X}_{\overline{\Theta}} = n(2n+1)[F : \mathbb{Q}_p]$ and in particular $\overline{X}_{\overline{\Theta}}$ contains a non-special point.

Proof. By Proposition 7.1 and Lemma 7.4 $\overline{X}_{\overline{\Theta}}$ identifies with the deformation functor of $\overline{\rho}$. From [Til96, Proposition 5.7] and the Euler characteristic formula [BJ19, Theorem 3.4.1], we know, that $\overline{X}_{\overline{\Theta}} \geq h^1(\Gamma_F, \mathfrak{sp}_{2n}) - h^2(\Gamma_F, \mathfrak{sp}_{2n}) = h^0(\Gamma_F, \mathfrak{sp}_{2n}) + n(2n+1)[F : \mathbb{Q}_p]$. By absolute irreducibility and Schur's lemma $h^0(\Gamma_F, \mathfrak{sp}_{2n}) = 0$. So from Proposition 8.11, we see, that the special locus $\overline{X}_{\overline{\Theta}}^{\mathrm{spcl}}$ is strictly contained in $\overline{X}_{\overline{\Theta}}$ and there must be a non-special point in $\overline{X}_{\overline{\Theta}}$. □

Remark 8.14. It is likely that the arguments of Section 8.3 carry over to $G = \mathrm{GSp}_{2n}$ with minor modifications. It is also likely that in future work we will be able to deduce the existence of non-special points for arbitrary residual Sp_{2n}- and GSp_{2n}-pseudocharacters, so that in Theorem 8.12 (3) equality holds.

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