Linearisations and projective GIT

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All schemes are assumed to be of finite type over an algebraically closed field k. For us a *variety* is a k-scheme, which is in addition integral and separated.

1 The ample cone

We complete talk 2 on the ample cone. We assume that X is a proper variety over k.

The reference for this section is [CLS11, §6].

Definition 1.0.1. The *Picard group* $\operatorname{Pic}(X)$ is the group of isomorphism classes of invertible $\mathcal{O}_X$-modules with $\otimes_{\mathcal{O}_X}$ as group multiplication.

Definition 1.0.2. We say that two invertible sheaves $\mathcal{L}_1$ and $\mathcal{L}_2$ on X are *algebraically equivalent*, if there exists a connected k-variety and $y_1, y_2 \in Y(k)$ and an invertible sheaf $\mathcal{L}$ on $X \times Y$, such that $\mathcal{L}|_{X_{y_i}} \cong \mathcal{L}_i$ for $i = 1, 2$. We define $\operatorname{Pic}^0(X)$ as the subgroup of invertible sheaves which are algebraically equivalent to $\mathcal{O}_X$. The quotient $\operatorname{NS}(X) := \operatorname{Pic}(X)/\operatorname{Pic}^0(X)$ is called the *Néron-Severi group* of X. We define $N^1_{\mathbb{R}}(X) := \operatorname{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

If X is a projective variety, then $\operatorname{Pic}(X)$ is the group of k-points of a separated and locally of finite type group scheme $\mathbf{Pic}(X)$ called the *Picard scheme* [Kle05, Theorem 4.8]. The subgroup $\operatorname{Pic}^0(X)$ is the group of k-points of the identity component $\mathbf{Pic}^0(X)$. If X is a normal proper variety, then $\mathbf{Pic}^0(X)$ is projective [Kle05, Proposition 5.4, Remark 5.6]. If $\operatorname{char}(k) = 0$, then $\mathbf{Pic}(X)$ is smooth and this is sufficient for $\mathbf{Pic}^0(X)$ to be an abelian variety. If $\operatorname{char}(k) > 0$, then there is a connected smooth projective surface X such that $\mathbf{Pic}(X)$ is not smooth, hence in this case $\mathbf{Pic}^0(X)$ is not an abelian variety [Kle05, Remark 5.15].

Theorem 1.0.3. If X is a normal proper variety, then $\operatorname{NS}(X)$ is a finitely generated abelian group.

Proof. [LN59, Theorem 2]. □

Example 1.0.4. If X is a normal proper toric variety, then $\operatorname{Pic}(X) = \operatorname{NS}(X)$. Indeed, if $\mathbf{Pic}^0(X)$ would be a nontrivial abelian variety, it would produce an uncountable supply of line bundles over any uncountable algebraically closed extension field of k, but for toric varieties we have seen that $\operatorname{Pic}(X)$ is countable.

If $\mathcal{L}$ is an ample invertible sheaf and algebraically equivalent to $\mathcal{L}'$, then $\mathcal{L}'$ is ample. So there is a well-defined notion of ample elements of $\operatorname{NS}(X)$.

Definition 1.0.5. The *ample cone* $\operatorname{Amp}(X)$ of X is the cone in $N^1_{\mathbb{R}}(X)$ generated by ample invertible sheaves in $\operatorname{NS}(X)$.

2 Projective GIT

The main reference for this part is [Hos12, §4]

2.1 Motivation

If we want to construct a GIT quotient of a projective k-scheme X, the following question comes to mind: Can we define reasonable quotients of X by gluing GIT quotients of G-stable affine open subschemes?

These quotients can be constructed, but the result might not be separated. This is illustrated by the following example.

Example 2.1.1. Let $X := \mathbb{A}^2 \setminus \{0\}$ equipped with the action of $\mathbb{G}_\mathrm{m} = \mathrm{Spec}(k[t, t^{-1}])$ by $t \cdot (x, y) := (tx, t^{-1}y)$. The subspaces $D(x)$, $D(y)$ and $D(xy)$ are $\mathbb{G}_\mathrm{m}$-stable and we can easily calculate their affine GIT quotients:

$$\begin{aligned} k[D(x)]^{\mathbb{G}_\mathrm{m}} &= k[x, y, x^{-1}]^{\mathbb{G}_\mathrm{m}} = k[xy] \\ k[D(y)]^{\mathbb{G}_\mathrm{m}} &= k[x, y, y^{-1}]^{\mathbb{G}_\mathrm{m}} = k[xy] \\ k[D(xy)]^{\mathbb{G}_\mathrm{m}} &= k[x, y, x^{-1}, y^{-1}]^{\mathbb{G}_\mathrm{m}} = k[xy, (xy)^{-1}] \end{aligned}$$

Gluing them together, we obtain $\mathbb{A}^1$ with a double origin, which is not separated!

Our strategy will thus be to take the Proj of invariants of a graded ring.

2.2 Recollections on projective geometry

We recall the basic features of the Proj construction. For every graded k-algebra A, which we will assume to be finitely generated, we can associate a scheme $\mathrm{Proj}(A)$ whose points are given by homogeneous prime ideals of A, which do not contain the *irrelevant ideal* $A_+ := \bigoplus_{n>0} A_n$. We call a k-scheme of the form $\mathrm{Proj}(A)$ *semiprojective*. When $A = k[x_0, \ldots, x_n]$, we have $\mathbb{P}^n = \mathrm{Proj}(A)$.

This construction is not a functor to k-schemes: If $f : A \to B$ is a map of graded k-algebras, we get a map of k-schemes $\mathrm{Proj}(B) \setminus V(f(A_+)) \to \mathrm{Proj}(A)$. When f is surjective, then $V(f(A_+)) = \emptyset$ and the map $\mathrm{Proj}(B) \to \mathrm{Proj}(A)$ is a closed immersion.

If X is a k-scheme and $\mathcal{L}$ is a line bundle on X, define for a section $s \in H^0(X, \mathcal{L})$ the set $U_s \subseteq X$ as the set of points $x \in X$, which have an open neighborhood V, such that $\mathcal{O}_V \to \mathcal{L}|_V,\ a \mapsto as$ is an isomorphism and $V_s := X \setminus U_s$. In particular s defines a trivialization of $\mathcal{L}$ on U_s. In that situation, we can see s as a local coordinate of X on U_s and we have a morphism $U_s \to \mathbb{A}^1$ defined by s. If $\bigcap_{s \in H^0(X,\mathcal{L})} V_s = \emptyset$, we say that $\mathcal{L}$ is *basepoint-free*.

Definition 2.2.1. An invertible sheaf $\mathcal{L}$ on X is *very ample*, if there exists some $n \geq 0$ and an immersion $\varphi : X \to \mathbb{P}^n$, such that $\mathcal{L} \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$. We say, that $\mathcal{L}$ is *ample* if there exists some $d \geq 1$, such that $\mathcal{L}^{\otimes d}$ is very ample.

X is projective if and only if X is proper and admits a very ample line bundle [Har77, Remark II.5.16.1].

We define the *ring of sections* of a line bundle $\mathcal{L}$ as

$$R(X, \mathcal{L}) := \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^{\otimes n})$$

If X is projective and $\mathcal{L}$ is an ample line bundle on X, then the induced map

$$X \to \mathrm{Proj}(R(X, \mathcal{L}))$$

is an isomorphism [Sta19, 0C6J].

2.3 Linearisation

Let G be a reductive group over k, which acts on a projective k-scheme X. Recall, that this action is given by a map of k-schemes

$$a : G \times X \to X$$

which satisfies the axioms of a group action in the category of k-schemes.

On $\mathbb{A}^{n+1}$ there is a natural action of GL_{n+1} which carries lines into lines. So this action descends to an action of PGL_{n+1} on $\mathbb{P}^n$. If X can be G-equivariantly embedded as a closed subscheme into $\mathbb{P}^n$, then we say that the action on X is *linearised* by the embedding. In algebraic terms such an action is especially simple: G acts linearly on the $n+1$-dimensional vector space spanned by $x_0, \dots, x_n$ and the action on $\mathbb{P}^n$ is given by the induced action on $\mathrm{Proj}(\mathrm{Sym}(V))$.

We now explain what we mean by a G-action on a line bundle.

Definition 2.3.1. Let $\mathcal{L}$ be a line bundle on X and assume that X is equipped with a G-action. A *G-linearisation* on $\mathcal{L}$ is an isomorphism

$$\Phi : a^*\mathcal{L} \to \mathrm{pr}_2^* \mathcal{L}$$

such that

$$(m \times \mathrm{id}_X)^*\Phi = \pi_{23}^*\Phi \circ (\mathrm{id}_G \circ a)^*\Phi$$

If $\mathcal{L}$ is (very) ample, we speak of a *(very) ample linearisation.*

The cocycle condition becomes more transparent when we look at elements of $G(k)$. For $g \in G(k)$ and $i_g : X \to G \times X,\ x \mapsto (g, x)$ we get an isomorphism $\varphi_g := i_g^*\Phi : g^*\mathcal{L} \to \mathcal{L}$. Using $(m \times \mathrm{id}_X) \circ (\mathrm{id}_G \times i_h) \circ i_g = i_{gh}$, we get

$$\varphi_{gh} = i_{gh}^*\Phi = i_h^*\Phi \circ h^* i_g^*\Phi = \varphi_h \circ h^*\varphi_g$$

for $g, h \in G(k)$.

Example 2.3.2.

1. A natural G-linearisation on $\mathbb{P}^n$ is given by the canonical bundle $\Lambda^n\Omega_{\mathbb{P}^n/k} = \mathcal{O}_{\mathbb{P}^n}(-(n+1))$ where the action on differential forms is given by $(g^{-1})^*$.
2. We will later be interested in the case of a linearisation on a trivial line bundle. Such a linearisation can be defined by a character $\chi : G \to \mathbb{G}_{\mathrm{m}}$. The map $\varphi_g^*\mathcal{O}_X \to \mathcal{O}_X$ can be defined by $\varphi_g(s)(x) := \chi(g)s(g^{-1}x)$. We denote by $\mathcal{L}_\chi$ the structure sheaf with G-action determined by a character χ.
3. Tensor products and inverses of linearisations carry a natural linearisation.

Theorem 2.3.3. Let G be a smooth connected linear algebraic group, which acts on a normal variety X. Then there is some $d \gg 0$, such that for all line bundles $\mathcal{L}$, there exists a G-linearisation on $\mathcal{L}^{\otimes d}$.

We give a variant of the argument in [Bri18, Theorem 5.2.1] only when X is projective. The following argument also does not show that d can be chosen independent of $\mathcal{L}$. However, if X is toric we could use that the Picard group is finitely generated (Theorem 1.0.3 and Example 1.0.4) to make d independent of $\mathcal{L}$.

Proof when X is projective. We want to use the following facts without proof: $\mathrm{Pic}(G)$ is finite [Bri18, Proposition 5.1.3] and $\mathrm{Pic}(G \times X) = \mathrm{Pic}(G) \times \mathrm{Pic}(X)$ [Bri18, Proposition 5.1.2].

We may assume that $\mathcal{L}$ is very ample. The pullback $a^*\mathcal{L}$ is isomorphic to $\pi_1^*\mathcal{M} \otimes \pi_2^*\mathcal{L}$ for some line bundle $\mathcal{M}$ on G. Replacing $\mathcal{L}$ with a high enough tensor power we may assume that $a^*\mathcal{L}$ and $\pi_2^*\mathcal{L}$ are isomorphic.

We choose any isomorphism $\Phi : a^*\mathcal{L} \cong \mathrm{pr}_2^*\mathcal{L}$. This provides us for every $g \in G(k)$ with an isomorphism $g^*\mathcal{L} \cong \mathcal{L}$. Since X is projective, such isomorphisms are unique up to a scalar in $H^0(X, g^*\mathcal{L} \otimes \mathcal{L}^*) = k$. So we obtain a k-homomorphism $G \to \mathrm{PGL}(H^0(X,\mathcal{L}))$, which induces an equivariant embedding $\phi : X \to \mathbb{P}(H^0(X,\mathcal{L}))$, such that $\mathcal{L} = \phi^*\mathcal{O}(1)$.

To make this argument precise in a scheme-theoretic way, we may argue as follows: Since the global sections of $G \times G \times X$ are $k[G \times G]$ and automorphisms of line bundles on $G \times G \times X$ are given by $k[G \times G]^\times$, we may define a family of scalars $\eta \in k[G \times G]^\times$ by the equation

$$(m \times \mathrm{id}_X)^*\Phi = \eta \cdot \pi_{23}^*\Phi \circ (\mathrm{id}_G \circ a)^*\Phi$$

to measure the failure of the cocycle condition. Using η we can write down a k-morphism $G \to \mathrm{GL}(H^0(X,\mathcal{L}))$, which is a homomorphism to $\mathrm{PGL}(H^0(X,\mathcal{L}))$. □

2.4 Projective GIT

Let G be a reductive group over k. If $\mathcal{L}$ is a linearisation of a semiprojective G-scheme X, then the action G on $R(X,\mathcal{L})$ preserves the grading. So $R(X,\mathcal{L})^G$ is a graded ring and is finitely generated by Nagata's theorem. Thus $\mathrm{Proj}(R(X,\mathcal{L})^G)$ is a semiprojective k-scheme. If X is projective, then $\mathrm{Proj}(R(X,\mathcal{L})^G)$ is projective. This leads to the following definition.

Definition 2.4.1. Let X be a semiprojective k-scheme with G-action and an ample linearisation $\mathcal{L}$. We define the GIT quotient of X by G with respect to $\mathcal{L}$ as

$$X /\!\!/_{\mathcal{L}} G := \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(X, \mathcal{L}^{\otimes n})^G\right)$$

The inclusion $R(X,\mathcal{L})^G \subseteq R(X,\mathcal{L})$ induces a rational map

$$X \dashrightarrow X /\!\!/_{\mathcal{L}} G$$

and we define the *semistable locus* X^{ss} as the complement of the null cone in X relative to this rational map.

Definition 2.4.2. A closed point $x \in X(k)$ is

1. *semistable*, if there is a section $s \in H^0(X, \mathcal{L}^{\otimes r})^G$ for some $r > 0$, such that $s(x) \neq 0$,
2. *stable*, if $\dim Gx = \dim G$ and there is a section s as in (1), such that and the action of G on $X_s = \{x \in X \mid s(x) \neq 0\}$ is closed,
3. *polystable*, if Gx is closed in X^{ss},
4. *unstable*, if x is not semistable.

Definition 2.4.3. [Hos12, Definition 2.36] A map $\varphi : X \to Y$ of finite type k-schemes where X is equipped with an action of a reductive group G is a *good quotient*, if the following hold.

1. φ is G-equivariant for the trivial action on the target.
2. φ is surjective.
3. If $U \subseteq Y$ is open, then $\mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$ is an isomorphism onto $\mathcal{O}_X(\varphi^{-1}(U))^G$
4. $W \subseteq X$ is a G-invariant closed subset, $\varphi(W)$ is closed in Y.
5. If W_1 and W_2 are disjoint G-invariant closed subsets of X, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint.

6. φ is affine.

Caution: Definition 2.4.3 does not coincide with Alper's definition [Alp09, Definition 4.1] of $[X/G] \to Y$ being a good moduli space. The condition of the pushforward along this map being exact cannot be satisfied in positive characteristic as taking invariants is not exact.

Theorem 2.4.4. $\pi_{\mathcal{L}} : X^{\mathrm{ss}} \to Y := X /\!\!/_{\mathcal{L}} G$ is a good quotient.

We follow the proof of [Hos12, Theorems 4.8, 4.11].

Proof. For $f \in R_+^G$ the localization $X_f^{\mathrm{ss}} = \mathrm{Spec}(R[\frac{1}{f}])$ is affine and $R^G[\frac{1}{f}] = R[\frac{1}{f}]^G$. Moreover the affine opens $Y_f = \mathrm{Spec}(R[\frac{1}{f}]^G)$ cover Y. The map $X_f^{\mathrm{ss}} \to Y_f$ is a good quotient, as we have seen in talk 3, see [Hos12, Theorems 3.14]. It follows, that $\pi_{\mathcal{L}}$ is a good quotient Zariski locally on the target. This implies that $\pi_{\mathcal{L}}$ is a good quotient. □

We denote by X^{ps} the set of polystable points in $X^{\mathrm{ss}}(k)$. We say, that two points $x, x' \in X^{\mathrm{ss}}$ are *S-equivalent*, if their orbit closures meet in X^{ss}.

Theorem 2.4.5. The map $X^{\mathrm{ps}}/G(k) \to (X /\!\!/_{\mathcal{L}} G)(k)$ induced by $\pi_{\mathcal{L}}$ is bijective.

We follow the proof of [Hos12, Theorems 4.30].

Proof. By the definition of polystable points the orbit closures of two points $x, x' \in X^{\mathrm{ps}}$ meet if and only if x and x' lie in the same orbit. We have seen before, that closed orbits biject with k-points of Y. □

2.5 Toric varieties as GIT quotients

The last goal of our talk is to show that toric varieties coming from polyhedra can be constructed as GIT quotients of an affine space by a torus action. The main reference is [CLS11, §14.2].

This time we start with a lattice M, which shall be identified with the character lattice of a torus T. Let P be a *polyhedron*, i.e. a convex set generated by a finite set in $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$.

Recall that the toric variety associated to P is given by

$$X_P := \mathrm{Proj}\left(\bigoplus_{n=0}^{\infty} \mathrm{span}_k(nP \cap M)\right) \tag{1}$$

Denote by $P(1)$ the set of *facets* of P, i.e. the set of codimension 1 faces. They correspond to dimension 1 subvarieties of X_P, which we will see as prime divisors D_F.

A facet F of P can be written as

$$F = \{m \in P \mid \langle m, u_F \rangle = -a_F\}$$

for some $u_F \in M^\vee$ and $a_F \in M_{\mathbb{R}}$.

The family of elements $(u_F)_{F \in P(1)}$ determines a homomorphism

$$M \to X^*(\mathbb{G}_{\mathrm{m}}^{P(1)}),\ m \mapsto ((t_F)_F \mapsto \prod_F t_F^{\langle m, u_F \rangle}).$$

Dually this determines a map of tori $\mathbb{G}_{\mathrm{m}}^{P(1)} \to T$ and we define a torus G as the kernel of this map. We have a left exact sequence

$$1 \to G \to \mathbb{G}_{\mathrm{m}}^{P(1)} \to T$$

which is also right exact if X_P has no torus factor, i.e. is not of the form $Y \times \mathbb{G}_{\mathrm{m}}^r$ for some toric variety Y and $r > 0$ [CLS11, Proposition 3.3.9]. We will assume now that X_P has no torus factor and identify M with a subgroup of $\mathbb{Z}^{P(1)}$, via $j : M \to \mathbb{Z}^{P(1)}$, $m \mapsto (\langle m, u_F \rangle)_F$.

For an arbitrary Weil divisor $D = \sum_F c_F D_F$ of X_P the polyhedron associated with D is

$$P_D = \{m \in P \mid \forall F \in P(1) : \langle m, u_F \rangle \geq -c_F\}$$

Further, D determines a homomorphism

$$\mathbb{G}_{\mathrm{m}}^{P(1)} \to \mathbb{G}_{\mathrm{m}}, \quad (t_F)_F \mapsto \prod_F t_F^{c_F}$$

and thereby a character $\theta_D : G \to \mathbb{G}_{\mathrm{m}}$.

In the following we will denote the GIT quotient with respect to $\mathcal{L}_{\theta_D}$ by $/\!\!/_D$.

Theorem 2.5.1. $\mathbb{A}^{P(1)} /\!\!/_D G = X_{P_D}$.

Further, if D is ample, then $X_{P_D} = X_P$.

Proof. The coordinate ring $\mathcal{O}(\mathbb{A}^{P(1)})$ is generated by variables x_F for each facet $F \in P(1)$ and the coordinate $g = (t_1, \ldots, t_{P(1)}) \in \mathbb{G}_{\mathrm{m}}^{P(1)}$ shall act on x_F by $g \cdot x_F := t_F^{-1} x_F$.

By definition

$$\mathbb{A}^{P(1)} /\!\!/_D G = \operatorname{Proj}\left(\bigoplus_{n=0}^{\infty} H^0(\mathbb{A}^{P(1)}, \mathcal{L}_{\theta_D}^{\otimes n})^G\right)$$

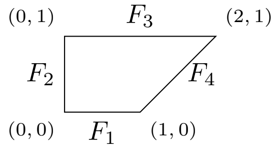
The line bundle $\mathcal{L}_{\theta_D}^{\otimes n}$ is trivial with G-action through θ_D^n. So we get twisted invariants $H^0(\mathbb{A}^{P(1)}, \mathcal{L}_{\theta_D}^{\otimes n})^G = k[x_1, \ldots, x_{P(1)}]_{\theta_D}^G$ in degree n, where $g \cdot x^{\mathbf{i}} = \theta_D^n(g) t^{-\mathbf{i}} x^{\mathbf{i}}$, where $\mathbf{i} \in \mathbb{N}_0^{P(1)}$. As a vector space $k[x_1, \ldots, x_{P(1)}]_{\theta_D}^G$ is spanned by monomials $x^{\mathbf{i}} = \prod_F x_F^{i_F}$, such that $n(c_F)_F - \mathbf{i} \in j(M)$, so we have

$$k[x_1, \ldots, x_{P(1)}]_{\theta_D}^G = \bigoplus_{\substack{m \in M \\ nc_F + \langle m, u_F \rangle \geq 0 \\ \forall F \in P(1) \geq 0}} k \cdot \prod_{F \in P(1)} x^{nc_F + \langle m, u_F \rangle} = \operatorname{span}_k(nP_D \cap M)$$

and this is precisely the degree n part of the homogeneous coordinate ring of X_{P_D} as in Equation (1). ☐

We close the talk with an example of Theorem 2.5.1.

Example 2.5.2 (Hirzebruch surface)**.** The Hirzebruch surface $X := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1))$ over $\mathbb{P}^1$ is isomorphic to X_P for the following polyhedron P in $\mathbb{R}^2$ and the lattice $M := \mathbb{Z}^2$.



The faces are given by

$$F_i = \{m \in P \mid \langle m, u_i \rangle \geq -a_i\}$$

where

$$u_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$a_1 = a_2 = 0, \quad a_3 = a_4 = -1.$$

The injection $M \hookrightarrow \mathbb{Z}^4$ is given by

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 1 \\ 1 & -1 \end{pmatrix}$$

The lattice M cuts out the closed subgroup G as the subtorus

$$G = \{(t_1, t_2, t_3, t_4) \in \mathbb{G}_\mathrm{m}^4 \mid t_2^{-1} t_4 = 1;\ t_1^{-1} t_3 t_4^{-1} = 1\}.$$

To each facet F_i we attach a T-invariant divisor $D_i := D_{F_i}$. Recall that there is a bijection between T-invariant divisors of codimension 1 and facets. The divisor $D_1 + D_2$ is ample, so $\mathbb{A}^4 /\!\!/_{D_1+D_2} G$ recovers X_P.

The quotient $\mathbb{A}^4 /\!\!/_{D_1} G$ is isomorphic to $\mathbb{P}^2$ and exhibits the map $X \to \mathbb{A}^4 /\!\!/_{D_1} G$ as a blowup of $\mathbb{P}^2$.

The quotient $\mathbb{A}^4 /\!\!/_{D_2} G$ is isomorphic to $\mathbb{P}^1$ and the map $X \to \mathbb{A}^4 /\!\!/_{D_2} G$ corresponds to the natural projection map.

References

[Alp09] Jarod Alper. *Good moduli spaces for Artin stacks*. 2009. arXiv: 0804.2242 [math.AG].

[Bri18] Michel Brion. *Linearisations of algebraic group actions*. = https://www-fourier.ujf-grenoble.fr/ mbrion/lin, 2018.

[CLS11] D.A. Cox, J.B. Little, and H.K. Schenck. *Toric Varieties*. Graduate studies in mathematics. American Mathematical Soc., 2011. ISBN: 9780821884263. URL: https://books.google.de/books?id=eXLGwYD4pmAC.

[Har77] Robin Hartshorne. *Algebraic geometry*. Graduate Texts in Mathematics, No. 52. New York: Springer-Verlag, 1977, pp. xvi+496. ISBN: 0-387-90244-9.

[Hos12] Victoria Hoskins. *Geometric invariant theory and symplectic quotients*. Available from the author's website: https://userpage.fu-berlin.de/hoskins/talks_and_notes.html. 2012.

[Kle05] Steven L. Kleiman. *The Picard scheme*. 2005. arXiv: math/0504020 [math.AG].

[LN59] S. Lang and A. Neron. "Rational Points of Abelian Varieties Over Function Fields". In: *American Journal of Mathematics* 81.1 (1959), pp. 95–118. ISSN: 00029327, 10806377. URL: http://www.jstor.org/stable/2372851 (visited on 10/28/2023).

[Sta19] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2019.