Animated commutative rings

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By *category* we mean an ordinary 1-category. By an ∞-*category* we mean an $(\infty, 1)$-category with a possibly large class of objects, but small simplicial sets as Hom spaces.

1 Animation

1.1 Sifted indization

Recall, that a nonempty simplicial set K is *sifted*, if the diagonal $K \to K \times K$ is cofinal [Lur09, Definition 5.5.8.1]. We refer to a colimit over a sifted diagram as a *sifted colimit*. We refer to the analogous 1-categorical notion as a 1-*sifted category*. The fundamental examples of sifted diagrams are:

1. $\mathrm{N}(\Delta^{\mathsf{op}})$ is sifted [Lur09, Lemma 5.5.8.4]. Colimits over it are called *geometric realizations*.
2. Every filtered diagram is sifted. [Lur09, Example 5.5.8.3]

Sifted colimits commute with finite products [Lur09, Lemma 5.5.8.11].

Definition 1.1. Let $\mathcal{C}$ be a small ∞-category which admits finite products and let $\mathcal{D}$ be an ∞-category. We denote by $\mathrm{Fun}^{\Pi}(\mathcal{C}, \mathcal{D})$ the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by functors which preserve finite products.

We denote by Ani the ∞-category of Kan complexes / ∞-groupoids / anima / spaces.

Definition 1.2. Let $\mathcal{C}$ be a small ∞-category with finite coproducts. We define the *sifted indization* as $\mathrm{sInd}(\mathcal{C}) := \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathsf{op}}, \mathrm{Ani})$.

Similarly, we define 1-sInd where we assume $\mathcal{C}$ to be an ordinary category and replace Ani by Set.

sInd is the free cocompletion under sifted colimits:

Proposition 1.3. Let $\mathcal{C}$ be a small ∞-category which admits finite coproducts and let $\mathcal{D}$ be an ∞-category which admits sifted colimits. Let $\mathrm{Fun}^{\Sigma}(\mathrm{sInd}(\mathcal{C}), \mathcal{D})$ denote the full subcategory of functors spanned by functors which preserve sifted colimits. Then the functor

$$\mathrm{Fun}^{\Sigma}(\mathrm{sInd}(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

induced by restriction along the Yoneda embedding $\mathcal{C} \to \mathrm{sInd}(\mathcal{C})$ is an equivalence of ∞-categories.

Proof. [Lur09, Proposition 5.5.8.15]. □

Lemma 1.4. Limits and sifted colimits in $\mathrm{sInd}(\mathcal{C})$ are computed objectwise. In particular, the inclusion $\mathrm{sInd}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ preserves and reflects limits and sifted colimits.

Proof. This follows, as limits and sifted colimits commute with finite products. □

In the presence of finite coproducts, sifted colimits are generated by filtered colimits and geometric realizations in the following sense:

Lemma 1.5. Let $\mathcal{C}$ be a small ∞-category with finite coproducts and let $X \in \mathrm{PSh}(\mathcal{C})$. The following are equivalent:

1. X belongs to $\mathrm{sInd}(\mathcal{C})$.
2. There is a simplicial object $U : \mathrm{N}(\Delta^{\mathrm{op}}) \to \mathrm{Ind}(\mathcal{C})$ whose colimit in $\mathrm{PSh}(\mathcal{C})$ is X.

Proof. This is [Lur09, Lemma 5.5.8.14]. □

The condition of having finite coproducts can probably not be relaxed, see [ARV10] for counterexamples in the 1-categorical setting.

Definition 1.6. An object X of an ∞-category $\mathcal{C}$ is *compact projective* if $\mathrm{Hom}_{\mathcal{C}}(X, -)$ preserves sifted colimits. We denote the full subcategory of compact projective objects by $\mathcal{C}^{\mathrm{cp}}$.

1.2 Animation

We recall basic properties of animation.

We say that an ordinary category $\mathcal{C}$ is 1-*compact projectively generated* if $\mathcal{C}$ admits small colimits and the natural functor

$$1\text{-}\mathrm{sInd}(\mathcal{C}^{\mathrm{cp}}) \to \mathcal{C}$$

is an equivalence. All 1-categories of interest to us will be 1-compact projectively generated.

For such categories it makes sense to give the following definition of a nonabelian derived category.

Definition 1.7. Let $\mathcal{C}$ be a 1-compact projectively generated 1-category. We define the *animation* $\mathrm{Ani}(\mathcal{C})$ as $\mathrm{Fun}^{\Pi}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Ani})$.

$\mathrm{Ani}(\mathcal{C})$ is presentable, [Lur09, Theorem 5.5.1.1, Proposition 5.5.8.10 (1)].

$\mathrm{Ani}(\mathcal{C})$ can also be defined as the localization of a model structure on simplicial objects of $\mathcal{C}$, [Lur09, Corollary 5.5.9.3].

Since truncation of anima commutes with finite products, we obtain functors $\pi_{\leq n} : \mathrm{Ani}(\mathcal{C}) \to \mathrm{Ani}(\mathcal{C})$ by applying $\pi_{\leq n}$ objectwise to anima. By [Lur09, Remark 5.5.8.26] this truncation coincides with the usual definition of truncation. We see directly that $\mathcal{C}$ can be identified with the 0-truncated objects of $\mathrm{Ani}(\mathcal{C})$.

There is a natural functor $\mathcal{C} = \mathrm{Fun}^{\Pi}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Set}) \to \mathrm{Fun}^{\Pi}((\mathcal{C}^{\mathrm{cp}})^{\mathrm{op}}, \mathrm{Ani}) = \mathrm{Ani}(\mathcal{C})$ induced by the functor $\mathrm{Set} \to \mathrm{Ani}$ which sends a set to the simplicial set it generates in degree 0.

Given a functor $F : \mathcal{C} \to \mathcal{D}$ of ordinary categories, which preserves sifted colimits, we have the following commutative diagram.

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{cp}} \arrow[r] \arrow[d] & \mathcal{D} \arrow[d] \\
\mathrm{Ani}(\mathcal{C}) \arrow[r] \arrow[d, "\pi_0"] & \mathrm{Ani}(\mathcal{D}) \arrow[d, "\pi_0"] \\
\mathcal{C} \arrow[r] & \mathcal{D}
\end{tikzcd}$$

The square where $\mathcal{C}^{\mathrm{cp}}$ in the upper left corner is replaced with $\mathcal{C}$ is in general not commutative! As a counterexample, one can consider $\mathcal{C} = \mathcal{D} = \mathrm{Ab}$ and a tensor product with a non-torsionfree abelian group.

Animation of functors does in general not commute with composition. We refer to [CS23, Proposition 5.1.5] for a criterion.

Lemma 1.8. Limits and filtered colimits of n-truncated objects of $\mathrm{Ani}(\mathcal{C})$ are n-truncated ($n \geq 0$).

Proof. This follows from Lemma 1.4 and the corresponding statement about anima. □

2 Animated commutative rings

By definition an *animated commutative ring* is an object of $\mathrm{Ani}(\mathrm{CRing}) = \mathrm{sInd}(\mathrm{CRing}^{\mathrm{cp}})$. However, unlike for sets, groups, abelian groups or modules, the category $\mathrm{CRing}^{\mathrm{cp}}$ is difficult to understand: $\mathrm{CRing}^{\mathrm{cp}}$ contains more than just polynomial rings. See MO 219938 for an example.

The following Lemma will allow us to work with a subcategory of $\mathcal{C}^{\mathrm{cp}}$, which will simplify proofs involving animation.

Lemma 2.1. Let $\mathcal{C}$ be a 1-compact projectively generated category and let $\mathcal{C}' \subseteq \mathcal{C}^{\mathrm{cp}}$ be a full subcategory, which generates $\mathcal{C}$ under sifted colimits. Then the induced functor $\mathrm{sInd}(\mathcal{C}') \to \mathrm{Ani}(\mathcal{C})$ is an equivalence.

Proof. Since $\mathcal{C}^{\mathrm{cp}}$ is a full subcategory of $\mathrm{Ani}(\mathcal{C})$, it follows that $\mathrm{Ani}(\mathcal{C})$ is generated under sifted colimits by $\mathcal{C}'$. The Lemma now follows from [Lur09, Proposition 5.5.8.22]. □

We define the 1-category Poly as the (small) 1-category with objects $\mathbb{Z}[x_1, \ldots, x_n]$ for $n \geq 0$, and ring homomorphisms as morphisms. Clearly Poly admits finite coproducts, consists of compact projective objects of CRing and generates CRing.

So, the natural functor

$$\mathrm{Fun}^{\Pi}(\mathrm{Poly}^{\mathsf{op}}, \mathrm{Set}) \to \mathrm{CRing}$$

is an equivalence.

Thanks to Lemma 2.1, we also have an equivalence

$$\mathrm{sInd}(\mathrm{Poly}) \simeq \mathrm{Ani}(\mathrm{CRing}).$$

3 Comparison with $\mathbb{E}_\infty$-rings

There is a natural functor

$$\Theta : \mathrm{Ani}(\mathrm{CRing}) \to \mathrm{CAlg}_{\mathbb{Z}/}$$

which is induced by the universal property of $\mathrm{sInd}(\mathrm{Poly}) = \mathrm{Ani}(\mathrm{CRing})$ applied to the natural inclusion functor $\mathrm{Poly} \to \mathrm{CAlg}_{\mathbb{Z}/}$.

Proposition 3.1. The functor Θ preserves small limits and colimits, is conservative and induces an equivalence $\mathrm{Ani}(\mathrm{CRing})_{\mathbb{Q}/} \to \mathrm{CAlg}_{\mathbb{Q}/}$.

Proof. [Lur18, Proposition 25.1.2.2]. □

Warning: The polynomial ring $\mathbb{Z}[x]$ does not map to a compact object in the $(\infty, 1)$-category of $\mathbb{E}_\infty$-rings.

4 Modules

Definition 4.1 (Modules, via animation)**.** We define Mod as the category of pairs (A, M), where A is a commutative ring and M is a module over it, with the obvious notion of morphisms. The functor $\mathrm{Mod} \to \mathrm{CRing}$, $(A, M) \mapsto M$ preserves sifted colimits and we can consider the animation $\mathrm{Ani(Mod)} \to \mathrm{Ani(CRing)}$. For $A \in \mathrm{Ani(CRing)}$, we define $\mathrm{Mod}_A^{\mathrm{ani}}$ as the fiber[1] over A.

As for Ani(CRing), we want to have a smaller system of compact projective generators.

Lemma 4.2. Pairs of the form $(\mathbb{Z}[x_1, \ldots, x_n], \mathbb{Z}[x_1, \ldots, x_n]^m)$ generate Mod under sifted colimits. We denote the full subcategory by PolyMod.

Proof. For a pair $(A, M) \in \mathrm{Mod}$, we first write A as a sifted colimit of polynomial rings. This reduces to the case that A is a polynomial ring. Then we write M as a sifted colimit of finite free A-modules. $\square$

If A is static, then $\mathrm{Mod}_A^{\mathrm{ani}} \simeq \mathrm{Ani}(\mathrm{Mod}_A^{\heartsuit})$, where $\mathrm{Mod}_A^{\heartsuit}$ is the category of classical A-modules.

We claim that the category of connective $\Theta(A)$-modules is canonically equivalent to the category of animated A-modules.

Proposition 4.3. $\mathrm{Mod}_{\Theta(A)}^{\mathrm{cn}} \simeq \mathrm{Mod}_A^{\mathrm{ani}}$.

Proof. This is explained in [Lur18, §25.2.1]. The idea is to define a category SCRMod, which consists of pairs (A, M), where $A \in \mathrm{Ani(CRing)}$ and M is a $\Theta(A)$-module. Then prove that the natural inclusion $\mathrm{PolyMod} \to \mathrm{SCRMod}$ induces an equivalence $\mathrm{Ani(Mod)} \simeq \mathrm{SCRMod}^{\mathrm{cn}}$. $\square$

In light of the above definition it makes sense to write $\mathrm{Mod}_A := \mathrm{Mod}_{\Theta(A)}$. We want to equip $\mathrm{Mod}_A^{\mathrm{ani}}$ with a derived tensor product, generalizing the derived functor of the tensor product in case A is static. There are two possible definitions:

Definition 4.4 (Tensor product, via $\mathbb{E}_\infty$-rings)**.** Let $A \in \mathrm{Ani(CRing)}$. Then we define $- \otimes_A - : \mathrm{Mod}_A \times \mathrm{Mod}_A \to \mathrm{Mod}_A$ as the tensor product of $\Theta(A)$-modules.

Definition 4.5 (Tensor product, via animation)**.** Let Mod^2 be the 1-category, which consists of triples (A, M_1, M_2), where A is a commutative ring and M_1, M_2 are A-modules. Then the usual tensor product defines a functor $\otimes : \mathrm{Mod}^2 \to \mathrm{Mod}$, $(A, M_1, M_2) \mapsto (A, M_1 \otimes_A M_2)$. We define the tensor product of animated rings as the animation

$$\mathrm{Ani}(\otimes) : \mathrm{Ani}(\mathrm{Mod}^2) \to \mathrm{Ani(Mod)}$$

These two definitions are actually equivalent. This comes down to checking that the tensor product functor $\mathrm{SCRMod}^2 \to \mathrm{SCRMod}$ in the operadic sense restricts to the usual tensor product on finite free objects.

As opposed to the situation for $\mathbb{E}_\infty$-rings, where we can just take a tensor product of modules, the tensor product of animated commutative rings must be defined separately. The idea is the same as in Definition 4.5.

For an ∞-category $\mathcal{C}$, we denote by $\mathrm{Span}(\mathcal{C})$ the category of *spans* of $\mathcal{C}$, that is diagrams of the form $y \leftarrow x \rightarrow z$. It can be defined as a functor category in the evident way.

Lemma 4.6. $\mathrm{Ani(Span(CRing))} \simeq \mathrm{Span(Ani(CRing))}$.

[1] Formally, we take the fiber product of $* \to \mathrm{Ani(CRing)} \leftarrow \mathrm{Ani(Mod)}$ in Cat_∞.

Proof. The idea is the same as in Lemma 4.2. By Lemma 2.1 we need to check that the finite free objects of Span(Ani(CRing)) generate it under sifted colimits. Given a span $(B \leftarrow A \to C)$, we first write A as a sifted colimit of polynomial rings, reducing to the case that A is a polynomial ring. We then write B as a sifted colimit of polynomial rings over A, and do the same for C. □

Definition 4.7. We define the tensor product functor

$$\otimes : \mathrm{Span}(\mathrm{CRing}) \to \mathrm{CRing}, \quad (B \leftarrow A \to C) \mapsto B \otimes_A C$$

The *tensor product* of animated rings is defined as the animation

$$\mathrm{Ani}(\otimes) : \mathrm{Ani}(\mathrm{Span}(\mathrm{CRing})) \to \mathrm{Ani}(\mathrm{CRing})$$

composed with the equivalence of Lemma 4.6.

We could also have defined the tensor product of animated commutative rings directly as a pushout:

Lemma 4.8. The tensor product $B \otimes_A C$ as in Definition 4.7 is a pushout in Ani(CRing).

Proof. Using Lemma 2.1, we need to check that the restriction of the pushout functor $\mathrm{Span}(\mathrm{Ani}(\mathrm{CRing})) \to \mathrm{Ani}(\mathrm{CRing})$ to the finite free objects is given by the usual tensor product. This is clearly the case. □

It follows from this Lemma and cocontinuity of Θ, that $\Theta(B \otimes_A C) \simeq \Theta(B) \otimes_{\Theta(A)} \Theta(C)$.

We close the discussion of tensor products by mentioning a useful spectral sequence for computations of tensor products:

Proposition 4.9. Given maps of animated commutative rings $B \leftarrow A \to C$ (or $\mathbb{E}_\infty$-rings) there is a convergent homological spectral sequence

$$E_2^{s,t} = \mathrm{Tor}_s^{\pi_*(A)}(\pi_*(B), \pi_*(C))_t \Rightarrow \pi_{s+t}(B \otimes_A C)$$

Proof. [Lur18, Remark 25.1.3.1]. □

5 Localization

We want to discuss localizations of animated rings in preparation for the construction of the derived spectrum of an animated ring in talk 5. One can deduce existence of general localizations quickly from the representability theorem, but we want to give a more concrete description of the standard Zariski localizations.

[Lur04, Example 3.4.8] [CS23, §5.1.7]

Definition 5.1. Let $A \in \mathrm{Ani}(\mathrm{CRing})$ and let $s \in \pi_0 A$. We define $A[s^{-1}] := A \otimes_{\mathbb{Z}[x]} \mathbb{Z}[x^{\pm 1}]$, where $x \mapsto s$.

Proposition 5.2. Let A and B be animated rings, let $s \in \pi_0 A$. Then the natural map $A \to A[s^{-1}]$ induces an equivalence of anima

$$\mathrm{Hom}^{s^{-1}}_{\mathrm{Ani}(\mathrm{CRing})}(A, B) \simeq \mathrm{Hom}_{\mathrm{Ani}(\mathrm{CRing})}(A[s^{-1}], B)$$

where $\mathrm{Hom}^{s^{-1}}_{\mathrm{Ani}(\mathrm{CRing})}$ denotes the full subanima of maps $f : A \to B$, such that $f(s)$ is invertible in $\pi_0 B$.

Proof. Define $\mathbb{Z}[x] \to A,\ x \mapsto s$. In the diagram

$$\begin{tikzcd}
\mathrm{Hom}^{s^{-1}}_{\mathrm{Ani(CRing)}}(A,B) \arrow[r] \arrow[d, dashed] \arrow[dd, bend right] & \mathrm{Hom}_{\mathrm{Ani(CRing)}}(A,B) \arrow[d] \\
\mathrm{Hom}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[x^{\pm 1}],B) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[x],B) \arrow[d] \\
\pi_0(B)^\times \arrow[r] & \pi_0(B)
\end{tikzcd}$$

the outer square is cartesian by definition. The lower square is cartesian, since

$$\mathrm{Hom}_{\mathrm{Ani(CRing)}}(\mathbb{Z}[x^{\pm 1}],B)$$

can be seen an as the fiber over zero of the map $B^2 \to B, (b,b') \mapsto bb'-1$. This can be seen by writing $\mathbb{Z}[x^{\pm 1}]$ as a pushout of $\mathbb{Z} \leftarrow \mathbb{Z}[z] \to \mathbb{Z}[x,y]$ mapping $z \mapsto 0$ and $z \mapsto xy-1$. The dotted arrow can then be defined by the universal property. We conclude that the upper square is cartesian. The proof is finished by Lemma 4.8. □

6 Derived symmetric power functors

The forgetful functors CRing $\to$ Ab $\to$ Set admit left adjoints. Via animation these promote to left adjoints of the forgetful functors

$$\mathrm{Ani(CRing)} \to \mathrm{Ani(Ab)} \to \mathrm{Ani}$$

The left adjoint $\mathrm{Ani(Sym)} : \mathrm{Ani(Ab)} \to \mathrm{Ani(CRing)}$ is called the *derived symmetric power functor*.

To ease the notation we will write Sym instead of Ani(Sym).

Example 6.1.

1. $\mathrm{Sym}(\mathbb{Z}^n) = \mathbb{Z}[x_1, \dots, x_n]$. This follows by composing the left adjoints above.
2. $\mathrm{Sym}(\mathbb{Z}[1]) = \mathbb{Z} \oplus \mathbb{Z}[1]$. This follows by writing $\mathbb{Z}[1]$ as a pushout, using that Sym is cocontinuous and computing the result as a derived tensor product of modules. The right hand side can be interpreted as the associated graded homotopy ring, in which multiplication of two elements in degree 1 is zero.

References

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